J.DIFFERENTIAL GEOMETRY
Vol. 42, No.3 November, 1995

SUBMANIFOLD GEOMETRY IN SYMMETRIC SPACES

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1. Introduction

The classical local invariants of a submanifold in a space form are the first fundamental form, the shape operators and the induced normal connection, and they determine the submanifold up to ambient isometry. One of the main topics in differential geometry is to study the relation between the local invariants and the global geometry and topology of submanifolds. Many remarkable results have been developed for submanifolds in space forms whose local invariants satisfy certain natural conditions. The study of focal points of a submanifold in an arbitrary Riemannian manifold arises from the Morse theory of the energy functional on the space of paths in the Riemannian manifold joining a fixed point to the submanifold. The Morse index theorem relates the geometry of a submanifold to the topology of this path space. The focal structure is intimately related to the local invariants of the submanifold. In the case of space forms one can go backwards and reconstruct the local invariants from the focal structure, so it is not too surprising that most of the structure theory of submanifolds can be reformulated in terms of their focal structure. What *is* perhaps surprising is a fact that became increasingly evident to the authors from their individual and joint research over the past decade: while extending the theory of submanifolds to ambient spaces more general than space-forms proves quite difficult if one tries to use the same approach as for the space forms, at least for symmetric spaces it has proved possible to develop an elegant theory based on focal structure that reduces to the classical theory in the case of space forms. This paper is an ex-

Received June 28, 1994. Research supported in part by NSF Grant DMS 9304285 for the first author, and by NSF Grant DMS 9100179 for the second author.

tended report on this theory, and the authors believe that the methods developed herein provide important tools for a continuing study of the submanifold geometry in symmetric spaces.

First we set up some notation. Let (N, g) be a Riemannian manifold, M an immersed submanifold of N, and $\nu(M)$ the normal bundle of M. The *end point map* $\eta : \nu(M) \to N$ of M is the restriction of the exponential map exp to $\nu(M)$. If $v \in \nu(M)_x$ is a singular point of η and the dimension of the kernel of $d\eta_v$ is m, then v is called a *multiplicity* m *focal normal* and $\exp(v)$ is called a *multiplicity* m *focal point* of M with respect to M in N. The *focal data*, $\Gamma(M)$, is defined to be the set of all pairs (v, m) such that v is a multiplicity m focal normal of M. The *focal variety* $\mathcal{V}(M)$ is the set of all pairs $(\eta(v), m)$ with $(v, m) \in \Gamma(M)$. The main purpose of our paper is to study the global geometry and topology of submanifolds in symmetric spaces whose focal data satisfy certain natural conditions.

In order to explain our results, we review some of the basic relations between focal points, Jacobi fields and Morse theory. For a fixed $p \in N$, let $P(N, M \times p)$ denote the space of H^1-paths $\gamma : [0,1] \to N$ such that $(\gamma(0), \gamma(1)) \in M \times \{p\}$ (a path is H^1 if it is absolutely continuous and the norm of its derivative is square integrable). Let

$$E : P(N, M \times p) \to R, \quad E(\gamma) = \int_0^1 \|\gamma'(t)\|^2 dt$$

be the energy functional. Then γ is a critical point of E if and only if γ is a geodesic normal to M at $\gamma(0)$ parameterized proportional to arc length. A vector field J along the geodesic γ is in the null space of the Hessian of E if and only if J satisfies the Jacobi equation

$$\nabla_{\gamma'} \nabla_{\gamma'} J - R(\gamma', J)(\gamma') = 0$$

with boundary conditions $J(0) \in TM_{\gamma(0)}$, $A_{\gamma'(0)}J(0) + J'(0) \in \nu(M)_{\gamma(0)}$ and $J(1) = 0$, where A_v is the shape operator of M with respect to the normal vector v. The Morse index theorem states that if p is not a focal point of M then E is a non-degenerate Morse function, and that the index of E at a critical point γ is the sum of the integers m such that $\gamma(t)$ is a multiplicity m focal point of M with respect to $\gamma(0)$ with $0 < t < 1$.

The basic local invariants of a submanifold are closely related to the structure of its focal variety. For given $v \in \nu(M)_x$, the tangent space

$T(\nu(M))_v$ can be naturally identified with $\nu(M)_x \oplus TM_x$. It is known that if $u \in TM_x$ then $d\eta_v(u) = J(1)$, where J is the Jacobi field on $\gamma(t) = \exp(tv)$ with the initial conditions $J(0) = u$ and $J'(0) = -A_v(u)$. Since the initial conditions of an ordinary differential equation determine the solution at time 1, the shape operators of M and the curvature tensor of N determine the focal structure of M, and one expects a close relation between the focal data, the local and global geometry, and the topology of the submanifold. For a general Riemannian manifold N, it is difficult to make this relation precise. But if N is a symmetric space, then the curvature tensor of N is a covariant constant. Hence in the coordinates obtained from a parallel normal frame along γ, the Jacobi equation becomes

$$(*) \qquad J'' + \mathcal{S}(J) = 0,$$

where $\mathcal{S}$ is a constant, self-adjoint operator whose eigenvalues can be expressed in terms of the roots of the symmetric space. This gives a precise relation between shape operators and focal data for submanifolds in symmetric spaces.

Recall that an *r-flat* in a rank k symmetric space $N = G/K$ is an r-dimensional, totally geodesic, flat submanifold. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition. Then every flat is contained in some k-flat, and every k-flat is of the form $\pi(g\exp(\mathfrak{a}))$, where $g \in G$ and $\mathfrak{a}$ is a maximal abelian subalgebra in $\mathfrak{p}$. If N is a compact Lie group of rank k, then a k-flat in N is just a maximal torus. But an r-flat need not be closed in general.

1.1. Definition. Let M be an immersed submanifold of a symmetric space N. The normal bundle $\nu(M)$ is called:

(i) *abelian* if $\exp(\nu(M)_x)$ is contained in some flat of N for each $x \in M$, and

(ii) *globally flat* if the induced normal connection is flat and has trivial holonomy.

Let v be a normal vector field on a submanifold M. The *end point map of* v is the map $\eta_v : M \to N$ defined by $x \mapsto \exp(v(x))$. If v is a parallel normal field v on M, then $M_v = \eta_v(M)$ is called the *parallel set defined by* v.

1.2. Definition. A connected, compact, immersed submanifold M in a symmetric space N is called *equifocal* if

(1) $\nu(M)$ is globally flat and abelian, and

(2) if v is a parallel normal field on M such that $\eta_v(x_0)$ is a multiplicity k focal point of M with respect to x_0, then $\eta_v(x)$ is a multiplicity k focal point of M with respect to x for all $x \in M$ (or equivalently, the focal data $\Gamma(M)$ is "invariant under normal parallel translation").

To simplify the terminology we make the following definition:

1.3. Definition. Let M be a submanifold in N, and $v \in \nu(M)_x$. Then t_0 is called a *focal radius* ($\frac{1}{t_0}$ is called a *focal curvature*) of M with multiplicity m along v if $\exp_x(t_0 v)$ is a multiplicity m focal point of M with respect to v.

Thus a submanifold M with globally flat abelian normal bundle of a symmetric space N is equifocal if the focal curvatures of M along any parallel normal field are constant.

A non-vanishing normal field v on M is called a *focal normal field* if $v/\parallel v \parallel$ is parallel and there exists m such that $v(x)$ is a multiplicity m focal normal of M for all $x \in M$. If v is a focal normal field, then $\parallel v \parallel$ is a smooth function on M, and the end point map $\eta_v : M \to N$ defined by $\eta_v(x) = \exp(v(x))$ has constant rank. So the kernel of $d\eta_v$ defines an integrable distribution $\mathcal{F}_v$ with $\eta_v^{-1}(y)$ as leaves, and $M_v = \eta_v(M)$ is an immersed submanifold of N. We will call $\mathcal{F}_v$, $\eta_v^{-1}(y)$ and M_v respectively the *focal distribution, focal leaf and the focal submanifold defined by the focal normal field v.*

1.4. Definition. A connected, compact, immersed submanifold M with a globally flat and abelian normal bundle in a symmetric space N is called *weakly equifocal* if given a parallel normal field v on M

(1) the multiplicities of focal radii along v are constant, i.e., the focal radius functions t_j are smooth functions on M and are ordered as follows:

$$\cdots < t_{-2}(x) < t_{-1}(x) < 0 < t_1(x) < t_2(x) < \cdots$$

and the multiplicities m_j of the focal radii $t_j(x)$ are constant on M,

(2) the focal radius t_j is constant on each focal leaf defined by the focal normal field $t_j v$ for all j, i.e., t_j is the pullback of a smooth function defined on the focal submanifold $M_{t_j v}$ via $\eta_{t_j v}$.

1.5. Remark. We will prove in section 5 that condition (2) on the focal radii in the definition of weakly equifocal submanifolds is always satisfied if the dimension of the focal distribution is at least two.

It follows from the definitions that a (weakly) equifocal submanifold in a rank-k symmetric space has codimension less than or equal to k, and that equifocal implies weakly equifocal.

When the ambient space N is the space form S^n, R^n or H^n, equifocal and weakly equifocal hypersurfaces have been extensively studied. Note that in this case, the operator S in the Jacobi equation (*) is cI, where c is the sectional curvature of the space form. It follows that t_0 is a focal curvature of multiplicity m along v if and only if $f_c(t_0)$ is a principal curvature along v of multiplicity m, where $f_c(t_0) = t_0$, $\cot(\frac{1}{t_0})$ or $\coth(\frac{1}{t_0})$ if $c = 0, 1$ or -1 respectively. So condition (2) for equifocal submanifolds and condition (1) for weakly equifocal submanifolds are equivalent to the conditions that the principal curvatures along any parallel normal field are constant and have constant multiplicities respectively. In these space forms they are referred to as *isoparametric* and *proper Dupin hypersurfaces* respectively. The study of isoparametric hypersurfaces in S^n has a long history, and these hypersurfaces have many remarkable properties (cf. [4], [24]). For example, assume that M is an isoparametric hypersurface of S^n with g distinct constant principal curvatures $\lambda_1 > \ldots > \lambda_g$ along the unit normal field v with multiplicities $m_1, \ldots, m_g$. Let E_j denote the curvature distribution defined by λ_j, i.e., $E_j(x)$ is equal to the eigenspace of $A_{v(x)}$ with respect to the eigenvalue $\lambda_j(x)$. Then the focal distributions of M are the curvature distributions E_j. It follows from the structure equations of S^n that there exists $0 < \theta < \frac{\pi}{g}$ such that the principal curvatures are $\lambda_j = \cot(\theta + \frac{(j-1)\pi}{g})$ with $j = 1, \ldots, g$, and the parallel set $M_t = M_{tv}$ for $-\frac{\pi}{g} + \theta < t < \theta$ is again an isoparametric hypersurface. The focal sets $M^+ = M_\theta$ and $M^- = M_{\theta - \frac{\pi}{g}}$ are embedded submanifolds of S^n with codimension $m_1 + 1$ and $m_g + 1$ respectively, and the focal variety of M in S^n is equal to

$$\{(x, m_1) \mid x \in M^+\} \cup \{(x, m_g) \mid x \in M^-\}.$$

Another consequence of the structure equations is that the leaves of each E_j are standard spheres. Using topological methods, Münzner proved the following:

(1) g has to be $1, 2, 3, 4$ or 6,

(2) $m_i = m_1$ if i is odd, and $m_i = m_2$ if i is even,

(3) S^n can be written as the union $D_1 \cup D_2$, where D_1 is the normal

disk bundle of M^+, D_2 is the normal disk bundle of M^- and $D_1 \cap D_2 = M$,

(4) the Z_2-homology of M can be given explicitly in terms of g and m_1, m_2; in particular, the sum of the Z_2-Betti numbers of M is $2g$.

Recall that a hypersurface M of S^n is called *proper Dupin* if the principal curvatures have constant multiplicities and $d\lambda(X) = 0$ for X in the eigenspace E_λ corresponding to the principal curvature λ (cf. [29]). Note that a hypersurface in S^n is proper Dupin if and only if it is weakly equifocal. It is proved in [38] that proper Dupin hypersurfaces have the above properties (1)-(4).

Recall also that a submanifold M in R^n is called *isoparametric* ([14], [5], [33]), if $\nu(M)$ is flat and the principal curvatures along any parallel normal field are constant. It is known that the normal bundle of a compact isoparametric submanifold in R^n is globally flat. So it follows that a compact submanifold in R^n is isoparametric if and only if it is equifocal. A submanifold M in R^n is called *weakly isoparametric* ([34]) if $\nu(M)$ is flat and the multiplicities of the principal curvatures λ along a parallel normal field v are constant, and if $d\lambda(X) = 0$ for X in the eigenspace $E_\lambda(v)$ corresponding to the principal curvature λ. It is proved in the papers quoted above that these submanifolds have many properties in common with isoparametric and proper Dupin hypersurfaces in spheres. One of the main goals of our paper is to generalize many of these results to equifocal and weakly equifocal submanifolds in compact symmetric spaces.

Henceforth, we will assume that $N = G/K$ is a compact, rank-k symmetric space of semi-simple type, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition, and N is equipped with the G-invariant metric given by the restriction of the negative of the Killing form of $\mathfrak{g}$ to $\mathfrak{p}$.

We first state a theorem that generalizes results on isoparametric hypersurfaces in spheres to equifocal hypersurfaces in compact symmetric spaces:

1.6. Theorem. *Let M be an immersed, compact, equifocal hypersurface in the simply connected, compact symmetric space N, and v a unit normal field. Then the following hold:*

(a) Normal geodesics are circles of constant length, which will be denoted by ℓ.

(b) *There exist integers* m_1, m_2, *an even number* $2g$ *and* $0 < \theta < \frac{\ell}{2g}$ *such that*

(1) *the focal points on the normal circle* $T_x = exp(\nu(M)_x)$ *are*

$$x(j) = exp\Big(\Big(\theta + \frac{(j-1)\ell}{2g}\Big)v(x)\Big), \qquad 1 \leq j \leq 2g,$$

and their multiplicities are m_1 *if* j *is odd and* m_2 *if* j *is even,*

(2) *the group generated by reflections in the pairs of focal points* $x(j), x(j+g)$ *on the normal circle* T_x *is isomorphic to the dihedral group* W *with* $2g$ *elements, and hence* W *acts on* T_x.

(c) M *is embedded.*

(d) $M \cap T_x = W \cdot x$.

(e) *Let* $\eta_{tv} : M \to N$ *denote the end point map defined by* tv, *and* $M_t = \eta_{tv}(M) = \{exp(tv(x)) \mid x \in M\}$ *denote the set parallel to* M *at distance* t. *Then* M_t *is an equifocal hypersurface and* η_{tv} *maps* M *diffeomorphically onto* M_t *if* $t \in (-\frac{\ell}{2g} + \theta, \theta)$.

(f) $M^+ = M_\theta$ *and* $M^- = M_{-\frac{\ell}{2g}+\theta}$ *are embedded submanifolds of codimension* $m_1 + 1, m_2 + 1$ *in* N, *and the maps* $\eta_{\theta v} : M \to M^+$ *and* $\eta_{(-\frac{\ell}{2g}+\theta)v} : M \to M^-$ *are* S^{m_1}- *and* S^{m_2}- *bundles respectively.*

(g) *The focal variety* $\mathcal{V}(M) = (M^+, m_1) \cup (M^-, m_2)$.

(h) $\{M_t \mid t \in [-\frac{\ell}{2g} + \theta, \theta]\}$ *gives a singular foliation of* N, *which is analogous to the orbit foliation of a cohomogeneity one isometric group action on* N.

(i) $N = D_1 \cup D_2$ *and* $D_1 \cap D_2 = M$, *where* D_1 *and* D_2 *are diffeomorphic to the normal disk bundles of* M^+ *and* M^- *respectively.*

(j) *Let* $p \in N$, $t \in R$, *and let* E *denote the energy functional on the path space* $P(N, p \times M_t)$. *Then the* Z_2*-homology of* $P(N, p \times M_t)$ *can be computed explicitly in terms of* m_1 *and* m_2 *and* t; *moreover,*

(1) *if* p *is not a focal point of* M *then* E *is a perfect Morse function,*

(2) *if* p *is a focal point of* M *then* E *is non-degenerate in the sense of Bott and perfect.*

Equifocal submanifolds, hyperpolar actions and infinite dimensional isoparametric submanifolds in Hilbert spaces are closely related as we will now explain. Recall that an isometric H-action on N is called *hyperpolar* if there exists a compact flat T in N, which meets every H-orbit and meets orthogonally at every point of intersection with an H-orbit (see [9], [27] and [17]). Such a T is called a *flat section* of the action. A typical example is the action of K on the compact rank-k symmetric space $N = G/K$ with k-flats in N as flat sections. It is proved by Bott and Samelson in [2] that this action is variationally complete, and that if M is an orbit in N, then $E : P(N, p \times M) \to R$ is a perfect Morse function and the Z_2-homology of $P(N, p \times M)$ can be computed explicitly in terms of the marked affine Dynkin diagram associated to the symmetric space N. We will prove in section 2 that principal orbits of a hyperpolar action are equifocal. Another main goals of our paper is to show that, although equifocal submanifolds in N need not be homogeneous, they share the same geometric and topological properties as principal orbits of hyperpolar actions. In particular, we will show that the Z_2-homology of $P(N, p\times M)$ can be similarly calculated if M is equifocal or more generally weakly equifocal.

Let $H^0([0,1], \mathfrak{g})$ denote the Hilbert space of L^2-integrable paths $u : [0,1] \to \mathfrak{g}$, and let $\phi : H^0([0,1], \mathfrak{g}) \to G$ be the *parallel transport map*, i.e., $\phi(u) = E(u)(1)$, where $E(u)$ satisfies the differential equation $E^{-1}E' = u$ with $E(0) = e$. We will see in section 4 that ϕ is a Riemannian submersion. It is proved in [37] that if M is a principal orbit of a hyperpolar action on G, then $\phi^{-1}(M)$ is isoparametric in $H^0([0,1], \mathfrak{g})$. We will show in section 5 that this statement is still true when M is equifocal, i.e., not assuming M is homogeneous.

Recall that a submanifold in R^n is called *taut* if all non-degenerate squared distance functions are perfect.

To summarize, we state some of our main results more precisely below.

1.7. Theorem. *If M is an immersed, weakly equifocal compact submanifold of a semi-simple, compact symmetric space N, then the following hold:*

(a) For a focal normal field v, the leaf of the focal distribution $\mathcal{F}_v$ through $x \in M$ is diffeomorphic to a taut submanifold of a finite dimensional Euclidean space.

(b) If $p \in N$ is not a focal point of M then $E : P(N, p \times M) \to R$

is a perfect Morse function, otherwise E is non-degenerate in the sense of Bott and perfect.

(c) *M is embedded in N.*

Recall that a submanifold M in N is called *totally focal* if $\eta^{-1}(\mathcal{V}(M)) = \Gamma(M)$ (cf. [6]). The next theorem is an analogue of Theorem 1.6 in higher codimension:

1.8. Theorem. *Suppose M is a codimension r equifocal submanifold of a simply connected, compact symmetric space N. Then th following hold:*

(a) *For a focal normal field v, the leaf of the focal distribution $\mathcal{F}_v$ through $x \in M$ is diffeomorphic to an isoparametric submanifold in $\nu(M_v)_{\eta_v(x)}$.*

(b) *$exp(\nu(M)_x) = T_x$ is an r-dimensional flat torus in N for all $x \in M$.*

(c) *There exists an affine Weyl group W with $r+1$ nodes in its affine Dynkin diagram such that for $x \in M$,*

  (1) W acts isometrically on $\nu(M)_x$, and the set of singular points of the W-action on $\nu(M)_x$ is the set of all $v \in \nu(M)_x$ such that $exp(v)$ is a focal point of M with respect to x,

  (2) $M \cap T_x = exp_x(W \cdot 0)$.

(d) *Let D_x denote the Weyl chamber of the W-action on $\nu(M)_x$ containing 0, and $\triangle_x = exp(D_x)$. Then:*

  (1) exp_x maps the closure $\overline{D_x}$ isometrically onto the closure $\overline{\triangle_x}$,

  (2) there is a labelling of the open faces of $\triangle_x$ by $\sigma_1(x), \dots, \sigma_{r+1}(x)$ and integers $m_1, \dots, m_{r+1}$ independent of x such that if $y \in \partial\triangle_x$, then y is a focal point with respect to x of multiplicity m_y, where m_y is the sum of m_i such that $y \in \overline{\sigma_i(x)}$.

(e) *M is totally focal in N.*

(f) *Let v be a parallel normal field on M. Then M_v is an embedded submanifold, and moreover:*

  (1) if $exp(v(x))$ is not a focal point, then M_v is again equifocal and the end point map $\eta_v : M \to M_v$ is a diffeomorphism,

(2) if $exp(v(x))$ is a focal point then $\eta_v : M \to M_v$ is a fibration and the fiber $\eta_v^{-1}(y)$ is diffeomorphic to a finite-dimensional isoparametric submanifold in the Euclidean space $\nu(M_v)_y$,

(3) $M_v \cap T_x = exp(W \cdot v(x))$.

(g) *Let x_o be a fixed point in M, and $\triangle = \triangle_{x_o}$. Then the following hold:*

(1) The parallel foliation $\{M_v \mid exp(tv(x_o)) \in \overline{\triangle}$ for all $0 \le t \le 1$ and v is a parallel normal field$\}$ is a singular foliation on N, which is analogous to the orbit foliation of a compact group action on N.

(2) Let $y \in \overline{\triangle}$ and M_y the parallel submanifold of M through y. Then the focal variety $\mathcal{V}(M) = \cup\{(M_y, m_y) \mid = y \in \partial\triangle\}$.

(h) *Let $p \in N$, v a parallel normal field on M, and E the energy functional on the path space $P(M, p\times M_v)$. Then the Z_2-homology of $P(M, p \times M_v)$ can be computed explicitly in terms of W and $m_1, \ldots, m_{r+1}$; moreover,*

(1) if p is not a focal point of M then E is a perfect Morse function,

(2) if p is a focal point of M then E is non-degenerate in the sense of Bott and perfect.

(i) *$N = \cup\{\overline{\triangle_x} \mid x \in M\}$, and*

(1) if $x \neq y$ then $\triangle_x \cap \triangle_y = \emptyset$,

(2) if $\overline{\triangle_x} \cap \overline{\triangle_y} \neq \emptyset$ then it is a closed subsimplex of both $\overline{\triangle_x}$ and $\overline{\triangle_y}$,

(3) given $x \in M$, $\{\triangle_y \mid y \in M \cap T_x\}$ is a triangulation of T_x.

Note that Theorem 1.8 (i) implies that we can associate to each equifocal submanifold of a simply connected, compact symmetric space N a "toric building structure on N". This is analogous to a spherical building except that the corresponding "apartments" cover tori instead of spheres.

Theorems 1.6 and 1.8 are not valid if N is not simply conneced. To see this, let N be the real projective space RP^n and M a distance sphere in N centered at x_0. Then M is certainly equifocal. Let v be a unit normal field on M. Then there exists $t_0 \in R$ such that $\exp(t_0 v(x)) = x_0$ for all $x \in M$. Let T_x be the normal circle at a point x in M. Then D_x is an interval, and $\triangle_x = T_x \setminus \{x_0\}$. Moreover, there exists t_1 such that the parallel set M_{t_1} is the cut locus of the center x_0, which is a Z_2-quotient of M, i.e., a projective hyperplane. Notice that the focal variety of M consists of only one point $(x_0, n-1)$, and M_{t_1} is not diffeomorphic to M. In fact M_{t_1} has the same dimension as M and satisfies all the conditions in the definition of an equifocal submanifold except that the normal bundle does not have trivial holonomy. Although a parallel manifold M_v of M in a simply connected symmetric space N is either equifocal or a focal submanifold, this need not be the case if N is not simply connected. In fact, in this case the parallel set of an equifocal submanifold in N is still an embedded submanifold, but there are three types of parallel submanifols. The third type is, as in the example just given, a parallel submanifold which satisfies all the conditions in the definition of an equifocal submanifold except that the normal bundle does not have trivial holonomy. This is analogous to the three types of orbits an action of a compact Lie group can have: principal, singular and exceptional orbits.

A submanifold M in N is called *curvature adapted* if for any $v \in \nu(M)_x$ the operator $B_v(u) = R(v,u)(v)$ leaves TM_x invariant, and B_v commutes with the shape operator A_v (cf. [3]). Wu in [41] defined a submanifold M in N to be *hyper-isoparametric* if it has globally flat abelian normal bundle, and is curvature adapted, and the principal curvatures along any parallel normal field are constant. Note that M is hyper-isoparametric if and only if M is curvature adapted and equifocal. Wu independently obtained some of our results by using the method of moving frames. But an equifocal submanifold in general is neither curvature adapted nor has constant principal curvatures. For example there are many such equifocal hypersurfaces in CP^n (cf. [40]).

We would like to make some comments on the methods we use to prove many of our geometric results on equifocal submanifolds. Since a symmetric space that is not a real space form has more than one root space, the operator S in the Jacobi equation (*) has more than one eigenvalue, and the shape operators and S need not commute in general. Thus there is no simple formula relating the focal points to the principal curvatures. This makes manipulation of the structure equations (the main technique in studying the geometry of submanifolds in space forms) much more complicated. So in this paper, we abandon many of the standard tools used in the study of submanifold geometry in space forms, and instead we study directly the relation between focal points of a submanifold and lifts of the submanifold under certain Riemannian submersions. In fact, the following two theorems are key steps in proving the results stated above:

1.9. Theorem. *Let $\pi : G \to G/K$ be the natural Riemannian fibration of the symmetric space G/K, M a submanifold of G/K with globally flat abelian normal bundle, and M^* a connected component of $\pi^{-1}(M)$. Let $v \in \nu(M)_x$, $x^* \in M^*$ $\pi(x^*) = x$, and v^* be the horizontal lift of v at x^*. Then:*

(1) $exp(v)$ is a multiplicity m focal point of M in G/K with respect to x if and only if $exp(v^*)$ is a multiplicity m focal point of M^* with respect to x^* in G,

(2) $\nu(M^*)$ is globally flat and abelian,

(3) M is equifocal (weakly equifocal resp.) in G/K if and only if M^* is equifocal (weakly equifocal resp.) in G.

1.10. Theorem. *Let $\phi : H^0([0,1], \mathfrak{g}) \to G$ denote the parallel transport map, M^* a compact submanifold in G with globally flat and abelian normal bundle, and $\tilde{M}$ a connected component of $\phi^{-1}(M^*)$. Let $v^* \in \nu(M^*)_{x^*}$, $\tilde{x} \in \tilde{M}$ $\phi(\tilde{x}) = x^*$, and $\tilde{v}$ be the horizontal lift of v^* at $\tilde{x}$. Then:*

(1) $exp(v^*)$ is a multiplicity m focal point of M^* in G with respect to x^* if and only if $exp(\tilde{v})$ is a multiplicity m focal point of $\tilde{M}$ with respect to $\tilde{x}$ in $H^0([0,1], \mathfrak{g})$,

(2) $\nu(\tilde{M})$ is globally flat,

(3) M^* is equifocal (weakly equifocal resp.) in G if and only if $\tilde{M}$ is isoparametric (weakly isoparametric resp.) in $H^0([0,1], \mathfrak{g})$.

So Theorems 1.9 and 1.10 allow us to study the geometry of an equifocal submanifold M in G/K by studying the geometry of the isoparametric submanifold $\tilde{M} = \phi^{-1}(\pi^{-1}(M))$ in the Hilbert space $V = H^0([0,1], \mathfrak{g})$. Although $\tilde{M}$ is infinite dimensional, the ambient space is a flat space form and most of the techniques and results in finite dimensional space forms are still valid (cf. [35], [28]).

This paper is organized as follows: We give examples of equifocal submanifolds in symmetric spaces in section 2, derive an explicit relation between focal points and shape operators of submanifolds in compact Lie groups in section 3, prove that the parallel transport map ϕ is a Riemannian submersion, and study the geometry of ϕ in section 4. We prove: Theorem 1.10 in section 5, Theorems 1.6, 1.7 and 1.8 and 1.9 in section 6, and the existence of inhomogeneous equifocal hypersurfaces in compact Lie groups and inhomogeneous isoparametri c hypersurfaces in Hilbert spaces in section 7. State some open problems are given in section 8.

The authors would like to thank the Max Planck Institute in Bonn. This research started during the authors' visit there in the fall of 1991 in an effort to find inhomogeneous isoparametric hypersurfaces in Hilbert spaces. It took us awhile to see how this simple problem actually ties together many interesting subjects in submanifold geometry.

2. Examples of equifocal submanifolds

The main result of this section is the following theorem.

2.1. Theorem. *Let H be a closed subgroup of $G \times G$ that acts on G isometrically by*

$$(h_1, h_2) \cdot g = h_1 g h_2^{-1}.$$

If the action of H on G is hyperpolar, then the principal H-orbits are equifocal submanifolds of G. More generally, if G/K is a compact symmetric space and H is a closed subgroup of G that acts hyperpolarly on G/K, then the principal H-orbits are equifocal.

Proof. First notice that if G/K is a compact symmetric space and H a closed subgroup of G which is hyperpolar on G/K, then the subgroup $H \times K$ of $G \times G$ is hyperpolar on G; see [17]. If $\pi : G \to G/K$ is the natural fibration, then the H-orbits in G/K lift to $H \times K$-orbits in G. We will show in section 6, that a submanifold of G/K is equifocal if and only if its lift to G is equifocal. It is therefore enough to prove the theorem for hyperpolar actions on a Lie group G.

Let M be a principal orbit in G. Then the induced action on the normal bundle of M is trivial, and every normal vector extends to an equivariant normal vector field. From the definition of hyperpolar actions it is clear that the normal bundle of M is abelian. It is proved in [27] that every equivariant normal field of M is parallel. Hence the normal bundle of M is globally flat.

It is therefore left to prove that if v is an equivariant normal field of M such that $\eta_v(x_0)$ is a multiplicity-k focal point of M with respect to x_0, then $\eta_v(x)$ is a multiplicity-k focal point of M with respect to x for all $x \in M$. This follows immediately from the following observations. First note that hyperpolar actions are variationally complete; see [9]. It is proved in [2](see also Proposition 2.7 in [36]) that variational completeness implies the following:

(i) The set of focal points of M in G is exactly the set of singular points with respect to the H-action.

(ii) If y is a multiplicity k focal point of M with respect to x, then k is equal to the difference of the dimensions of the isotropy subgroups, i.e., $k = \dim(H_y) - \dim(H_x)$.

2.2. Remark. Let M be a principal orbit of an isometric H-action on G. It is proved in [17] that if $\exp(\nu(M)_x)$ is contained in some flat, then the H-action is hyperpolar. In particular, this implies that if the normal bundle of M is abelian then M is equifocal.

2.3. Examples. The following are examples of hyperpolar actions on a Lie group G or a symmetric space G/K (cf. [17]):

(a) $H = G(\sigma) = \{(g, \sigma(g)) \mid g \in G\}$, where $\sigma : G \to G$ is an automorphism, is hyperpolar on G.

(b) $H = K_1 \times K_2$ is hyperpolar on G, where K_i is the fixed point set of some involution σ_i of G. Consequently K_1 is hyperpolar on the symmetric space G/K_2.

(c) The action of $H = \rho(K) \times SO(n-1)$ on $SO(n)$, where $\rho : K \to SO(n)$ is the isotropy representation of a rank-two symmetric space.

(d) any cohomogeneity one action on G/K.

2.4. Remark. It is true in real space forms that a hypersurface is equifocal if and only if it has constant principal curvatures. This is not true in more general ambient spaces as an example of Wang [40] shows. In his example Wang starts with an inhomogeneous isoparametric hypersurface in an odd dimensional sphere which he shows to be the lift under the Hopf map of a hypersurface M in a complex projective space. This hypersurface M is equifocal in our terminology. He then shows that the principal curvatures of M cannot be constant. Since a curvature adapted, equifocal hypersurface must have constant principal curvatures, these hypersurfaces are not curvature adapted. They are not homogeneous either because their lifts are not homogeneous. We will discuss related examples of inhomogeneous equifocal hypersurfaces in the last section of this paper.

3. Relation between focal points and shape operators

Let G be a compact, semi-simple Lie group, $\mathfrak{g}$ its Lie algebra, $\langle\,,\,\rangle$ an Ad-invariant inner product on $\mathfrak{g}$, and ds^2 the bi-invariant metric on G defined by $\langle\,,\,\rangle$. The main result of this section is to give a necessary and sufficient condition for a point in G to be a focal point of a submanifold with abelian and globally flat normal bundle in G.

It is known that the Levi-Civita connection for ds^2 is given by

$$\nabla_X Y = \frac{1}{2}[X,Y]$$

if X, Y are left invariant vector fields, and

$$\nabla_X Y = -\frac{1}{2}[X,Y]$$

if X, Y are right invariant vector fields. The curvature tensor is

$$R(X,Y)Z = \frac{1}{4}[X,[Y,Z]],$$

where X, Y, and Z are left invariant vector fields.

To simplify the notation, we will use the convention that for $g \in G$ and $y \in \mathfrak{g}$, $gy = (L_g)_*(y)$ and $yg = (R_g)_*(y)$, where L_g and R_g are left and right translations by g respectively.

3.1. Proposition. *If M is a submanifold in G with abelian normal bundle, then*

$$[x^{-1}\nu(M)_x, \mathfrak{g}] \subset x^{-1}TM_x.$$

In particular, $R(\xi, TG_x)\xi \subset TM_x$ for $\xi \in \nu(M)_x$.

Proof. Set $\mathfrak{A} = x^{-1}\nu(M)_x$. Since the inner product on $\mathfrak{g}$ is Ad-invariant and $\mathfrak{A}$ is abelian, we have

$$\langle[\mathfrak{g},\mathfrak{A}],\mathfrak{A}\rangle = \langle\mathfrak{g},[\mathfrak{A},\mathfrak{A}]\rangle = \langle\mathfrak{g},0\rangle = 0,$$

i.e., $[\mathfrak{g},\mathfrak{A}] \subset x^{-1}TM_x$. The second part of the proposition follows since

$$R(\xi,X)\xi = \frac{1}{4}[\xi,[X,\xi]].$$

The following is an elementary fact concerning focal points and Jacobi fields.

3.2. Proposition. *Let M be a submanifold of N, $x(t)$ a smooth curve in M, $v(t)$ a normal field of M along $x(t)$, and $\eta : \nu(M) \to N$ the end point map. Then*

$$d\eta_{v(0)}(v'(0)) = J(1),$$

where J is the Jacobi field along $x(t)$ satisfying the initial condition $J(0) = x'(0)$ and $J'(0) = -A_{v(0)}(x'(0)) + \nabla^{\perp}_{x'(0)}v$, where $\nabla^{\perp}$ is the normal connection of the submanifold M.

Let $a \in \mathfrak{g}$, $\gamma(t) = xe^{at}$ be a geodesic in G, and $J(t)$ be a Jacobi field along γ. Denote the parallel transport map along γ from $\gamma(t_1)$ to $\gamma(t_2)$ by $P_\gamma(t_1, t_2)$. Set $Y(t) = \gamma(0)^{-1}P_\gamma(t,0)J(t)$. Then the Jacobi equation for J gives rise to the following equation for Y (cf. [23]):

$$Y'' - \frac{1}{4}\,\mathrm{ad}(a)^2 Y = 0.$$

Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{g}$ that contains a, and

$$\mathfrak{g} = \mathfrak{t} + \sum_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$$

the root space decomposition with respect to $\mathfrak{t}$, where $\mathrm{ad}(a)^2(z_\alpha) = -\alpha(a)^2 z_\alpha$ for $z_\alpha \in \mathfrak{g}_\alpha$. Let $D_1(a)$ and $D_2(a)$ be the operators defined as follows: for $z = p_0 + \sum_\alpha p_\alpha$ with $p_0 \in \mathfrak{t}, p_\alpha \in \mathfrak{g}_\alpha$,

$$D_1(a)(z) = p_0 + \textstyle\sum_\alpha \cos\left(\frac{\alpha(a)}{2}\right)p_\alpha,$$
$$D_2(a)(z) = p_0 + \textstyle\sum_\alpha \frac{2}{\alpha(a)}\sin\left(\frac{\alpha(a)}{2}\right)p_\alpha,$$

where $\lambda^{-1}\sin(\lambda)$ is defined to be 1 if $\lambda = 0$. Notice that $D_1(a)$ and $D_2(a)$ depend only on a, but not on the choice of the maximal abelian subalgebra $\mathfrak{t}$ containing a. One can also describe D_1 and D_2 in terms of the curvature tensor of G. For this we note that the operator, $R_a(z) = R(a,z)a = -\frac{1}{4}\,\mathrm{ad}(a)^2(z)$, is a nonnegative symmetric operator, and

$$D_1(a)(z) = \cos(\sqrt{R_a}(z)),$$
$$D_2(a)(z) = \sqrt{R_a^{-1}}(z)\sin(\sqrt{R_a}(z)).$$

3.3. Theorem. *Suppose M is a submanifold in G with abelian normal bundle and $a \in x^{-1}\nu(M)_x$. Then*

(1) the operator $(D_1(a) - D_2(a)x^{-1}A_{xa}x)$ maps $x^{-1}TM_x$ to itself, where A_{xa} is the shape operator of M at x along xa,

(2) xe^a is a focal point of M of multiplicity m with respect to x if and only if the operator $(D_1(a) - D_2(a)x^{-1}A_{xa}x)$ on $x^{-1}TM_x$ is singular with nullity m.

Part (1) of this theorem is obvious, but to prove part (2) we need the following Lemma.

3.4. Lemma. *Let $x(t)$ be a curve in M, and $v(t) = x(t)a(t)$ a parallel normal field along $x(t)$. Then*

$$x_0^{-1}P_{\gamma_0}(1,0)d\eta_{(x_0,v_0)}(v_0') = \{D_1(a) - D_2(a)x_0^{-1}A_{v_0}x_0\}(x_0^{-1}x_0'),$$

where 0 as an index refers to $t = 0$ and $P_{\gamma_0}(1,0)$ denotes the parallel transport map along the geodesic $\gamma_0(s) = x_0e^{sa_0}$ from $\gamma_0(1)$ to $\gamma_0(0)$.

Proof. Let $\gamma(s,t) = x(t)e^{sa(t)}$ be a variation of normal geodesics of M, and

$$S = \frac{\partial\gamma}{\partial s}, \quad T = \frac{\partial\gamma}{\partial t}.$$

Then $S(0,t) = x(t)a(t)$, and $J(s) = T(s,0)$ is a Jacobi field along the geodesic $\gamma_0(s) = x(0)e^{sa(0)}$ with $J(0) = x'(0)$. By Proposition 3.2, we have

$$d\eta_{v(0)}(v'(0)) = J(1),$$

and $J'(0) = -A_{v(0)}(x'(0))$ since $v(t)$ is parallel. As above set

$$Y(s) = \gamma_0(0)^{-1}P_{\gamma_0}(s,0)J(s).$$

Then Y satisfies the differential equation

$$Y'' - \frac{1}{4}\,\mathrm{ad}(a)^2 Y = 0.$$

Since the operator $\mathrm{ad}(a)^2$ is in the diagonal form with respect to the root space decompositon, this differential equation can be solved explicitly. In fact, the solution for the initial value problem $Y(0) = p_0 + \sum_\alpha p_\alpha$ and $Y'(0) = q_0 + \sum_\alpha q_\alpha$ is

$$Y(t) = p_0 + tq_0 + \sum_\alpha p_\alpha \cos\Big(\frac{\alpha(a)}{2}t\Big) + q_\alpha \frac{2}{\alpha(a)} \sin\Big(\frac{\alpha(a)}{2}t\Big).$$

Clearly

$$Y'(0) = x(0)^{-1}J'(0) = -x(0)^{-1}A_{v(0)}(x'(0)).$$

But $Y(0) = x(0)^{-1}x'(0)$, so we have

$$\begin{aligned} x_0^{-1}P_{\gamma_0}(1,0)d\eta_{v_0}(v_0') &= Y(1) \\ &= p_0 + q_0 + \textstyle\sum_\alpha p_\alpha \cos\left(\frac{\alpha(a)}{2}\right) + q_\alpha \frac{2}{\alpha(a)} \sin\left(\frac{\alpha(a)}{2}\right) \\ &= \{D_1(a) - D_2(a)x_0^{-1}A_{v_0}x_0\}(x_0^{-1}x_0'), \end{aligned}$$

where 0 as an index on x, x' v and v' refers to $t = 0$.

3.5. Proof of Theorem 3.3. It is obvious that we can choose a basis for $T(\nu(M))_{(x,xa)}$, which consists of vectors of the form $v'(0)$ as in Lemma 3.4 and $\sigma'(0)$ with $\sigma(t) = x(a+tb)$, $b \in x^{-1}\nu(M)_x$. Since $\nu(M)$ is abelian, we have $d\exp_{xa}(xb) = xe^a b \neq 0$. The theorem now follows from Lemma 3.4.

For $a \in \nu(M)_x$, $\nu(M)_x \oplus TM_x$ can be naturally identified with $T(\nu(M))_a$ via the map

$$(b,u) \mapsto v_b'(0) + v_u'(0),$$

where v_b is defined by $v_b(s) = a + sb$ and $v_u(s)$ is the parallel normal field along the geodesic $\gamma(s) = \exp^M(su)$ with $v_u(0) = a$ (here $\exp^M$ denotes the exponential map of M). It follows from the proof of Theorem 3.3 that the differential $d\eta : T(\nu(M))_a \to TG_{\eta(a)}$ is of the form

$$\begin{pmatrix} I & 0 \\ 0 & D_1(a) - D_2(a)A_a \end{pmatrix}$$

under the identifications

$$\begin{aligned} T(\nu(M))_a &\simeq \nu(M)_x \oplus TM_x, \\ TG_{\eta(a)} &\simeq \nu(M)_x \oplus TM_x, \end{aligned}$$

where in the second line $(b,u) \in \nu(M)_x \oplus TM_x$ is identified with $P_{\gamma_0}(0,1)(b+u)$ and γ_0 is defined by $\gamma_0(s) = \exp(sa)$. In particular, this implies that the kernel of $d\eta_a$ is a subset of TM_x under the the above identifications (notice that we have very strongly used that the normal bundle $\nu(M)$ is abelian). So we have:

3.6. Proposition. *Let M be a submanifold of G with abelian and globally flat normal bundle, and v a parallel normal vector field. Then*

$y_0 = \exp v(x_0)$ is a focal point of M with respect to x_0 if and only if the differential of the end point map of v, $\eta_v : M \to G, x \to \exp v(x)$, is not injective at x_0. Moreover, the multiplicity of y_0 as a focal point is equal to the dimension of the kernel of $d(\eta_v)_{x_0}$.

Theorem 3.3 is valid for any symmetric space. In fact, let N be a symmetric space, $G = \mathrm{Iso}(N)$, and M a submanifold of N with abelian normal bundle. Let $x_0 \in M$, $K = G_{x_0}$, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the Cartan decomposition, and $a \in \mathfrak{p}$ normal to M at x_0. Let $\mathfrak{A}$ be a maximal abelian subalgebra in $\mathfrak{p}$ containing a, and

$$\mathfrak{p} = \mathfrak{A} + \sum_{\alpha \in \triangle} \mathfrak{p}_\alpha$$

its root space decomposition. Define $D_i(a) : \mathfrak{p} \to \mathfrak{p}$ by $D_i(a)(b) = b$ for $b \in \mathfrak{A}$, and

$$D_1(a)(x_\alpha) = \cos\left(\frac{\alpha(a)}{2}\right) x_\alpha, \quad D_2(a)(x_\alpha) = \frac{2}{\alpha(a)} \sin\left(\frac{\alpha(a)}{2}\right) x_\alpha,$$

for $x_\alpha \in \mathfrak{p}_\alpha$. Then the same proof as for Theorem 3.3 implies that $\exp(a)$ is a multiplicity m focal point of M with respect to x_0 if and only if the kernel of the operator $D_1(a) - D_2(a)A_a$ is of dimension m. So we have the same result as in Proposition 3.6 for symmetric spaces:

3.7. Proposition. *Let M be a submanifold of a symmetric space N with abelian and globally flat normal bundle, and v a parallel normal vector field on M. Then $y_0 = \exp v(x_0)$ is a focal point of M with respect to x_0 if and only if the differential of the end point map of v, $\eta_v : M \to N, x \to \exp v(x)$, is not injective at x_0. Moreover, the multiplicity of y_0 as a focal point is equal to the dimension of the kernel of $d(\eta_v)_{x_0}$.*

The following Proposition will be useful later.

3.8. Proposition. *Let M be a submanifold in G with globally flat and abelian normal bundle, $x(t) \in M$, and $a(t) \in \mathfrak{g}$ such that $v(t) = x(t)a(t)$ is a normal vector field of M along the curve $x(t)$. Then*

(1) $x(0)^{-1}\nabla_{x'(0)} v = a'(0) + 1/2[x(0)^{-1}x'(0), a(0)]$,

(2) $v(t)$ is parallel if and only if $a'(0) \in x(0)^{-1}TM_{x(0)}$. Furthermore, if $v(t)$ is parallel, then

$$-x(0)^{-1}A_{v(0)}(x'(0)) = a'(0) + \frac{1}{2}[x(0)^{-1}x'(0), a(0)].$$

Proof. Let $y_1, \ldots, y_n$ be a basis for $\mathfrak{g}$, and $Y_i(g) = gy_i$ the left invariant vector field defined by y_i. Write $a(t) = \sum_i = f_i(t)y_i$. Then $v(t) = \sum_i f_i(t)Y_i(x(t))$. Since the metric on G is bi-invariant, we have

$$\begin{aligned}\nabla_{x'(0)}v &= \textstyle\sum_i f_i'(0)Y_i(x(0)) + f_i(0)(\nabla_{x'(0)}Y_i)(0)\\ &= \textstyle\sum_i f_i'(0)x(0)y_i + \frac{1}{2}f_i(0)x(0)[x(0)^{-1}x'(0), y_i]\\ &= x(0)\{a'(0) + \tfrac{1}{2}[x(0)^{-1}x'(0), a(0)]\}.\end{aligned}$$

This proves (1), and (2) follows easily from (1).

4. The geometry of the parallel transport map

Let $H^0([0,1];\mathfrak{g})$ denote the Hilbert space of L^2-integrable paths $u : [0,1] \to \mathfrak{g}$, where the L^2-inner product is defined by

$$\langle u, v\rangle_0 = \int_0^1 \langle u(t), v(t)\rangle\, dt.$$

Let $H^1([0,1];G)$ denote the group of absolutely continuous paths $g : [0,1] \to G$ such that g' is square integrable, i.e., $\langle g'g^{-1}, g'g^{-1}\rangle_0$ is finite. Given a subset H of $G \times G$, we let $P(G,H)$ denote the subset of all $g \in H^1([0,1],G)$ such that $(g(0), g(1)) \in H$. If H is a subgroup, then $P(G,H)$ is a subgroup of $H^1([0,1],G)$. Note that $P(G, e\times G)$ is a Hilbert manifold, and

$$T(P(G, e\times G))_g = \{vg \mid v \in H^1([0,1];\mathfrak{g}),\ v(0) = 0\}.$$

Let $P(G, e\times G)$ be equipped with the right-invariant metric defined by

$$\langle v_1 g, v_2 g\rangle = \langle v_1', v_2'\rangle_0.$$

Let $E : H^0([0,1];\mathfrak{g}) \to P(G, e \times G)$ be the parallel translation in the trivial principal bundle $I \times G$ over $I = [0,1]$ defined by the connection $u(t)dt$, i.e., $E = E(u)$ for $u \in H^0([0,1];\mathfrak{g})$ is the unique solution of

$$\begin{cases} E^{-1}E' &= u,\\ E(0) &= e.\end{cases}$$

Let $\phi : H^0([0,1];\mathfrak{g}) \to G$ be the *parallel transport from* 0 *to* 1, i.e.,

$$\phi(u) = E(u)(1).$$

The following result is known (cf. [35], [37]):

4.1. Theorem. *Let $H^1([0,1],G)$ act on $V = H^0([0,1];\mathfrak{g})$ by gauge transformations, i.e.,*

$$g * u = gug^{-1} - g'g^{-1},$$

and let $\phi : H^0([0,1],\mathfrak{g}) \to G$ be the parallel transport map from 0 to 1. Then the foloowing hold:

(1) The action of $H^1([0,1],G)$ on $H^0([0,1];\mathfrak{g})$ is proper, Fredholm and isometric.

(2) $\phi(g * u) = g(0)\phi(u)g^{-1}(1)$ for $g \in H^1([0,1],G)$ and $u \in H^0([0,1];\mathfrak{g})$.

(3) If $\phi(u) = x_0\phi(v)x_1^{-1}$ then there exists $g \in H^1([0,1],G)$ such that $g(0) = x_0, g(1) = x_1$ and $u = g * v$.

4.2. Corollary. *The action of $P(G, e\times G)$ on $H^0([0,1];\mathfrak{g})$ by gauge transformations is transitive and free.*

Proof. Let $\hat{y}$ denote the constant path with constant value y. We first prove that the action is transitive. Let $u \in H^0([0,1];\mathfrak{g})$. Since G is connected and compact, there exists $y \in \mathfrak{g}$ such that $\phi(u) = e^y$. It is obvious that $\phi(\hat{0})e^y = \phi(u)$. So by Theorem 4.1, there exists $g \in P(G, e \times G)$ with $g(0) = e$ and $g(1) = e^{-y}$ such that $u = g * \hat{0}$. To prove the action is free, let $g \in P(G, e \times G)$ be such that $g * \hat{0} = \hat{0}$. Then $g^{-1}g' = 0$. So g is constant. Since $g(0) = e$, $g(t) = e$ for all t.

4.3. Corollary. *The parallel translation $E : H^0([0,1],\mathfrak{g}) \to P(G, e\times G)$ is an isometry.*

Proof. Since the inverse of E is the map $F : P(G, e \times G) \to H^0([0,1],\mathfrak{g})$ defined by $F(g) = g^{-1}g' = g^{-1} * \hat{0}$ and $dF_g(vg) = g^{-1}v'g$, our claim follows.

Let G be a compact Lie group acting on a smooth manifold M, and let $\pi : M \to M/G$ denote the orbit space map. It is known that if the action of G has only one orbit type, then the orbit space M/G has a unique differentiable structure such that π is a fibration. If moreover, M is Riemannian and the G-action is isometric, then there exists a unique metric on M/G such that π is a Riemannian submersion. In fact, we have $T(M/G)_{\pi(x)} = d\pi_x(TM_x)$ and $\langle d\pi_x(u_1), d\pi_x(u_2)\rangle_{\pi(x)} = \langle u_1, u_2\rangle_x$ for u_1, $u_2 \in (\ker d\pi_x)^\perp$ and $x \in M$. In the following we will give an infinite dimensional analogue of this fact for the free isometric action of $\Omega_e(G) = P(G, e \times e)$ on $H^0([0,1];\mathfrak{g})$. First, as a consequence of Theorem 4.1 and Corollary 4.2 we have

4.4. Corollary. *Let $\Omega_e(G)$ denote the subgroup $P(G, e\times e)$ of $P(G, e\times G)$, and $\phi : V = H^0([0,1];\mathfrak{g}) \to G$ the parallel transport map from 0 to 1. Then:*

(1) the gauge action of $\Omega_e(G)$ on V is free,

(2) the fibers of ϕ are exactly the orbits of $\Omega_e(G)$,

(3) ϕ is a principal $\Omega_e(G)$-bundle,

(4) G is the orbit space $V/\Omega_e(G)$, and ϕ is the orbit space map,

(5) for $g \in P(G, e\times G)$ the map $\tau_g(u) = g * u$ maps fibers of ϕ to fibers of ϕ,

(6) the differential of the map τ_g in (5) is $d(\tau_g)_u(v) = gvg^{-1}$.

4.5. Theorem. *Let $\mathcal{H}$ be the horizontal distribution for the fibration ϕ, i.e., $\mathcal{H}(u)$ be the normal space of the fiber $\phi^{-1}(\phi(u))$ at u, and let $\hat{y}$ denote the constant path in $\mathfrak{g}$ with constant value y. Then the following:*

(1) $\mathcal{H}(\hat{0}) = \{\hat{y} \mid y \in \mathfrak{g}\}$,

(2) if $u = g * \hat{0}$, then $\mathcal{H}(g * \hat{0}) = g\mathcal{H}(\hat{0})g^{-1} = \{g\hat{y}g^{-1} \mid y \in \mathfrak{g}\}$,

(3) if $u = g * \hat{0}$ with $g(0) = e$, then $d\phi_u(g\hat{y}g^{-1}) = y\phi(u) = yg(1)^{-1}$,

(4) ϕ is a Riemannian submersion,

(5) ϕ is the natural Riemannian submersion associated to the free isometric action of $\Omega_e(G)$ on $H^0([0,1],\mathfrak{g})$.

Proof. Since $F = \Omega_e(G) * \hat{0} = \{-g'g^{-1} \mid g \in \Omega_e(G)\}$, we have

$$\begin{aligned} TF_{\hat{0}} &= \{\xi' \mid \xi \in H^1([0,1];\mathfrak{g}),\ \xi(0) = \xi(1) = 0\} \\ &= \{u \in H^0([0,1];\mathfrak{g}) \mid \textstyle\int_0^1 u(t)dt = 0\}. \end{aligned}$$

So (1) is an immediate consequence. Statement (2) follows from the facts that the map $\tau_g(v) = g*v$ is an isometry on $H^0([0,1];\mathfrak{g})$, $d(\tau_g)_{\hat{0}}(v) = gvg^{-1}$, and τ_g maps fibers of ϕ at 0 to fibers of ϕ at $g*0$. Note that if $g(0) = e$ and $u = g * \hat{0}$ then

$$\begin{aligned} \phi(g * \hat{0} + sg\hat{y}g^{-1}) &= \phi(g * \hat{sy}), \quad \text{by Theorem 4.1} \\ &= \phi(\hat{sy})g(1)^{-1} = e^{sy}g(1)^{-1} = e^{sy}\phi(u). \end{aligned}$$

So $d\phi_u(g\hat{y}g^{-1}) = y\phi(u)$, which proves (3). Then (4) and (5) follows.

4.6. Corollary. *Let $v \in TG_x$, $u \in \phi^{-1}(x)$, and $\tilde{v}$ be the horizontal lift of v at u with respect to ϕ. Choose $g \in P(G, e\times G)$ such that $u = g * \hat{0}$. Then*

$$\tilde{v}(u) = gvx^{-1}g^{-1}.$$

Let M be a submanifold in G. Using the isometry E from $H^0([0,1],\mathfrak{g})$ to $P(G, e\times G)$, we see that $\phi^{-1}(M)$ is isometric to the submanifold

$$P(G, e\times M) = \{g\in P(G, e\times G)\mid g(0)=e, g(1)\in M\}$$

of $P(G, e\times G)$. Note that $P(G, e\times M)$ is like a cylinder set for the Wiener measure on the set of continuous paths g in G. Motivated by the definiton of a general cylinder set, we consider below the parallel transport from a to b for any $0\le a<b\le 1$. Let $H^0([a,b],\mathfrak{g})$ denote the space of L^2-paths in $\mathfrak{g}$ with the inner product defined by

$$\langle u, v\rangle = \int_a^b \langle u(t), v(t)\rangle\, dt.$$

Let

$$\phi_a^b : H^0([a,b],\mathfrak{g}) \to G$$

denote the parallel transport map from a to b for the connections $u(t)dt$ over $[a,b]$, i.e., ϕ_a^b is defined by $\phi_a^b(u)=g(b)$, where $g:[a,b]\to G$ is the solution of the initial value problem:

$$g^{-1}g' = u, \quad g(a)=e.$$

The proof of the following facts is the same as for $\phi_0^1=\phi$.

4.7. Theorem. *Let ds^2 be a fixed bi-invariant metric on G, and let $G_{[a,b]}$ denote the Lie group G with the bi-invariant metric $\frac{1}{b-a}\,ds^2$, and $\phi_a^b : H^0([a,b],\mathfrak{g})\to G_{[a,b]}$ the parallel transport map from a to b. Then the following hold:*

(1) $\phi_a^b(g*u) = g(a)\phi_a^b(u)g(b)^{-1}$,

(2) if $\phi_a^b(v) = x_1\phi_a^b(u)x_2^{-1}$ then there exists $g\in H^1([a,b],G)$ such that $g(a)=x_1$, $g(b)=x_2$ and $v=g*u$,

(3) the horizontal space of ϕ_a^b at $g*0$ is $\{g\hat{y}g^{-1}\mid y\in\mathfrak{g}\}$,

(4) if $u=g*\hat{0}$, then $d(\phi_a^b)_u(g\hat{y}g^{-1}) = (b-a)y\phi_a^b(u)$,

(5) ϕ_a^b is a Riemannian submersion, and it is the natural Riemannian submersion associated to the free, isometric action of

$$\Omega_e([a,b],G) = \{g\in H^1([a,b],G)\mid g(a)=g(b)=e\}$$

on $H^0([a,b],\mathfrak{g})$.

4.8. Corollary. *Let* $\underline{s}: 0 = s_0 < s_1 \cdots < s_n = 1$ *be a partition of* $[0,1]$, *and*

$$\Phi_{\underline{s}} : H^0([0,1];\mathfrak{g}) \to \textstyle\prod_{i=1}^{i=n} G_{[s_{i-1},s_i]}, \quad \textit{defined by}$$
$$\Phi_{\underline{s}}(u) = (\phi_0^{s_1}(u \mid [0,s_1]), \dots, \phi_{s_{n-1}}^1(u \mid [s_{n-1},1])),$$

and identify $H^0([0,1],\mathfrak{g})$ *with the direct sum* $\oplus_{i=1}^n H^0([s_{i-1},s_i],\mathfrak{g})$ *via the linear isometry*

$$f : H^0([0,1],\mathfrak{g}) \to \bigoplus_{i=1}^n H^0([s_{i-1},s_i],\mathfrak{g}) \quad \textit{defined by}$$
$$u \to (u \mid [0,s_1], \dots, u \mid [s_{n-1},1]).$$

Then the following hold:

(1) $\Phi_{\underline{s}}$ *is the natural Riemannian submersion associated to the free product action of* $\prod_{i=1}^n \Omega_e([s_{i-1},s_i],G)$ *on* $H^0([0,1],\mathfrak{g})$,

(2) $\Phi_{\underline{s}}(u)$
$= (E(u)(s_1), E(u)(s_1)^{-1}E(u)(s_2), \dots, E(u)(s_{n-1})^{-1}E(u)(1)).$

(3) *if* M *is a submanifold of* $\prod_i G_{[s_{i-1},s_i]}$ *then* $\Phi_{\underline{s}}^{-1}(M)$ *is isometric to the following submanifold of* $P(G, e \times G)$:
$\{g \in P(G, e\times G) \mid (g(s_1), g(s_1)^{-1}g(s_2), \dots, g(s_{n-1})^{-1}g(1)) \in M\}.$

Proof. Let $[a,b] \subset [0,1]$. By the uniqueness of solution of ordinary differential equations we have $\phi_a^b(u \mid [a,b]) = E(u)(a)^{-1}E(u)(b)$. So the Corollary follows.

5. The geometry of lifts of submanifolds of G to $H^0([0,1];\mathfrak{g})$

Let M be a submanifold of a Hilbert space V, and $\eta : \nu(M) \to V$ the end point map (i.e., $\eta(v) = x + v$ if $v \in \nu(M)_x$). Recall that M is called *proper Fredholm* if the restriction of the end point map to any normal disk bundle $\nu_r(M)$ of finite radius r is a proper, Fredholm map. If M is proper Fredholm in V then $I - A_v$ is Fredholm for all $v \in \nu(M)$, and the restriction of any squared distance function to M satisfies condition C of Palais and Smale. A proper Fredholm submanifold M of V is called *isoparametric* if $\nu(M)$ is globally flat and for any parallel normal field v on M the shape operators $A_{v(x)}$ and $A_{v(y)}$ are conjugate for all $x, y \in M$.

We refer to [35] and [28] for more detailed geometric and topological properties of these submanifolds.

A proper Fredholm submanifold M of V is called *weakly isoparametric* if $\nu(M)$ is globally flat, the multiplicities of the principal curvatures along a parallel normal field v are constant, and $d\lambda(X) = 0$ for $X \in E_\lambda(v) = \{X \mid A_v X = \lambda X\}$ where λ is a principal curvature function λ along a parallel normal field v.

Let G be a compact, connected, semi-simple Lie group equipped with a bi-invariant metric, and H a closed subgroup of $G \times G$ acting on G by $(h_1, h_2) \cdot x = h_1 x h_2^{-1}$. Let $P(G,H)$ act on $H^0([0,1]; \mathfrak{g})$ by gauge transformations. It is proved in [37] that if the action of H on G is hyperpolar, then the action of $P(G,H)$ on $H^0([0,1]; \mathfrak{g})$ is polar and the principal $P(G,H)$-orbits are isoparametric. Furthermore, using the formula $\phi(g * u) = g(0)\phi(u)g(1)^{-1}$ and the fact that the fibers of ϕ are orbits of $\Omega_e(G)$, we obtain $\phi^{-1}(H \cdot e^a) = P(G,H) * \hat{a}$. To summarize, we have

5.1. Theorem ([37]). *If M is a principal orbit of a hyperpolar action on G, then $\phi^{-1}(M)$ is an isoparametric submanifold in $H^0([0,1], \mathfrak{g})$.*

By Theorem 2.1, a principal orbit of a hyperpolar action is equifocal. So Theorem 1.10 (3) in the introduction generalizes Theorem 5.1 to equifocal submanifolds M in Lie groups which are not necessarily homogeneous.

Before starting with the proof of Theorem 1.10 we give a few simple applications. Recall that a proper Fredholm submanifold M of a Hilbert space V is *taut* (cf. [35]) if for every non-focal point $a \in V$ the distance squared function $f_a : M \to R$ defined by $f_a(x) = \| x - a \|^2$ is a perfect Morse function.

5.2. Proposition. *Let M be a weakly equifocal immersion of a compact manifold into a compact Lie group G. Then the lift $\tilde{M}$ of M to V is taut.*

Proof. By Theorem 1.10, $\tilde{M}$ is weakly isoparametric. One can prove exacly as in the finite dimensional case(see [34], [35] and [38]) that an infinite dimensional weakly isoparametric submanifold is taut.

5.3. Proposition. *A weakly equifocal immersion of a compact manifold M into a Lie group G is an embedding.*

Proof. We know from Proposition 5.2 that $\tilde{M}$ is taut. It is standard that taut submanifolds are embedded (cf. [35]). Hence $\tilde{M}$ is embedded and then it is clear that M is embedded as well.

5.4. Proposition. *Let M be a weakly equifocal compact submanifold in G. If $p \in G$ is not a focal point of M, then the energy functional $E : P(G, p \times M) \to R$ is a perfect Morse function.*

Proof. Set $p = e^a$ for $a \in \mathfrak{g}$. Consider the diffeomorphism $\rho : P(G, p \times M) \to P(G, e \times M)$ defined by $\rho(g)(t) = g(t)e^{(t-1)a}$. The path space $P(G, p \times M)$ can be naturally embedded into $H^0([0,1], \mathfrak{g})$ as the submanifold $\tilde{M} = \phi^{-1}(M)$ via the map $g \mapsto F(\rho(g))$ where $F : P(G, e \times G) \mapsto H^0([0,1], \mathfrak{g})$ is the isometry defined by $F(g) = g^{-1}g'$ as in the proof of Corollary 4.3. Using this embedding, the functional E on $P(G, p \times M)$ corresponds to the restriction of the Hilbert distance squared function $f_a(u) = \| u - a \|_0^2$ to $\tilde{M}$. Since $\tilde{M}$ is taut (Proposition 5.2), f_a is a perfect Morse function.

We will now prove several lemmas needed for the proof of Theorem 1.10. The notation will be the same as in Theorem 1.10 except that we do not assume that M has a globally flat normal bundle when not explicitly stated. First as a consequence of Corollary 4.6, we have

5.5. Lemma. *Suppose $h \in P(G, e \times G)$, $u = h * \hat{0} \in \tilde{M}$ and $x = \phi(u)$. Then*

$$\nu(\tilde{M})_u = \{hbx^{-1}h^{-1} \mid b \in \nu(M)_x\}.$$

5.6. Lemma. *Let v be a normal vector field on M, and $\tilde{v}$ the horizontal lift of v to $\tilde{M}$. Then $\tilde{v}$ is a parallel normal field on $\tilde{M}$ if and only if v is parallel on M.*

Proof. We need to show that $d\tilde{v}_u(T\tilde{M}_u)$ is contained in $T\tilde{M}_u$ if and only if $dv_x(TM_x) \subset TM_x$. Now let $h \in P(G, e \times G)$ be such that $u = h * \hat{0}$, and g_s a smooth curve in $P(G, e \times G)$ such that $g_0(t) = e$ for all t and $g_s * u \in \tilde{M}$. Then $x(s) = \phi(g_s * u) \in M$. Let

$$\xi = \frac{d}{ds}\Big|_{s=0} g_s, \qquad \xi(u) = \frac{d}{ds}\Big|_{s=0} (g_s * u), \quad a(x) = x^{-1}v(x).$$

A direct computation gives

$$d\tilde{v}_u(\xi(u)) = \tfrac{d}{ds}\Big|_{s=0} \tilde{v}(g_s * h * 0)$$
$$= \tfrac{d}{ds}\Big|_{s=0} g_s hx(s)a(x(s))x(s)^{-1}h^{-1}g_s^{-1}$$
$$= [\xi, hx_0a(x_0)x_0^{-1}h^{-1}] + h[x'(0)x_0^{-1}, x_0a(x_0)x_0^{-1}]h^{-1} + hx_0a'(0)x_0^{-1}h^{-1},$$

where $x_0 = x(0)$. By Lemma 5.5, $\nu(\tilde{M})_u = \{hx_0bx_0^{-1}h \mid x_0b \in \nu(M)_{x_0}\}$. Given $b \in x_0^{-1}\nu(M)_{x_0}$, because $x_0^{-1}\nu(M)_{x_0}$ is abelian and $\langle,\rangle$ is ad-invariant, we have

$$\langle[\xi, hx_0a(x_0)x_0^{-1}h^{-1}] + h[x'(0)x_0^{-1}, x_0a(x_0)x_0^{-1}]h^{-1}, hx_0bx_0^{-1}h^{-1}\rangle_0 = 0.$$

By Proposition 3.8 (2), v is parallel if and only if $a'(0) \in x_0^{-1}TM_{x_0}$. Hence v is parallel if and only if

$$\langle hx_0a'(0)x_0^{-1}h^{-1}, hx_0bx_0^{-1}h^{-1}\rangle_0 = 0,$$

which holds if and only if $\tilde{v}$ is parallel. This completes the proof of the lemma.

As a consequence of Proposition 3.8 and the proof of Lemma 5.6 we have:

5.7. Lemma. *The shape operators A_v and $\tilde{A}_{\tilde{v}}$ are related as follows:*

$$-h^{-1}\tilde{A}_{\tilde{v}}(\xi(u))h = [h^{-1}\xi h, vx_0^{-1}] + \frac{1}{2}[x'(0)x_0^{-1}, vx_0^{-1}] - A_v(x'(0))x_0^{-1}.$$

5.8. Lemma. *Let M be a compact submanifold of G. Then $\tilde{M} = \phi^{-1}(M)$ is a proper Fredholm submanifold of $V = H^0([0,1];\mathfrak{g})$. Furthermore, the shape operators of $\tilde{M}$ are compact.*

Proof . We prove properness first. Suppose $\xi_k \in \nu(\tilde{M})_{u_k}$, $\|\xi_k\| \le r$, and $u_k + \xi_k \to w$. We want to prove that ξ_k has a convergent subsequence. To prove this, we choose $g_k \in P(G, e \times G)$ such that $g_k * 0 = u_k$. Let $x_k = \phi(u_k)$, and $a_k \in x_k^{-1}\nu(M)_{x_k}$ be such that ξ_k is the horizontal lift of x_ka_k, i.e.,

$$\xi_k = g_kx_ka_kx_k^{-1}g_k^{-1}.$$

Compactness of M implies that there exist a subsequence of x_k (still denoted by x_k) and $x_0 \in M$ such that $x_k \to x_0$. Since the disk of radius r in $\mathfrak{g}$ is compact, by passing to a subsequence we may assume that $x_ka_kx_k^{-1} \to b \in \nu(M)_{x_0}x_0^{-1}$. But

$$u_k + \xi_k = -g_k'g_k^{-1} + g_k(x_ka_kx_k^{-1})g_k^{-1} = g_k * (x_ka_kx_k^{-1}) \to w$$

and $x_ka_kx_k^{-1} \to b$. Since the action of $P(G, e \times G)$ on $H^0([0,1];\mathfrak{g})$ is proper (cf. [35]), g_k has a subsequence in $H^1([0,1]; G)$ converging to g_0. This implies that $u_k \to u_0 = g_0 * 0 \in \tilde{M}$. Hence ξ_k converges.

To prove $\tilde{M}$ is Fredholm, we will show that its shape operators are compact. Using right translation if necessary, we may assume that $e \in M$. Let $a \in \nu(M)_e$ be a non-zero normal vector. It suffices to prove that the shape operator $\tilde{A}_{\hat{a}}$ at $\hat{0} \in \tilde{M}$ is a compact operator. Since $\xi(\hat{0}) = -\xi'$, using Lemma 5.7 we get

$$\tilde{A}_{\hat{a}}(\xi') = [\xi, a] + \frac{1}{2}[x'(0), a] - A_a(x'(0)).$$

We can thus write

$$\tilde{A}_{\hat{a}}(\xi') = D(\xi') + B(\xi'),$$

where

$$D(\xi') = [\xi, a] \quad \textit{and} \quad B(\xi') = \frac{1}{2}[x'(0), a] - A_a(x'(0)).$$

It is clear that B is of finite rank. So it suffices to prove that D is compact. To see this, we let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{g}$ containing $\nu(M)_e$, $\mathfrak{g} = \mathfrak{t} + \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ the corresponding root space decomposition, and $\{t_1, \dots, t_k\} \cup \{x_\alpha, y_\alpha \mid \alpha \in \Delta^+\}$ an orthonormal basis of $\mathfrak{g}$ such that $t_{k-p+1}, \dots, t_k \in \nu(M)_e$, $t_j \in \mathfrak{t}$, $x_\alpha, y_\alpha \in \mathfrak{g}_\alpha$ and

$$[a, x_\alpha] = \alpha(a) y_\alpha, \quad [a, y_\alpha] = -\alpha(a) x_\alpha.$$

Let $z_\alpha = x_\alpha + i y_\alpha$. Then

$$\begin{aligned} e_{\alpha,n} &= Re(z_\alpha e^{2\pi i n t}), \quad f_{\alpha,n} = Im(z_\alpha e^{2\pi n t}), \quad \alpha \in \Delta^+, \quad n \geq 1, \\ s_{j,n} &= t_j \cos 2\pi n t, \quad t_{j,n} = t_j \sin 2\pi n t, \quad n \geq 1, \\ & t_1, \dots, t_{k-p}, \quad x_\alpha, y_\alpha, \quad \alpha \in \Delta^+ \end{aligned}$$

form an orthonormal basis for $T\tilde{M}_{\hat{0}}$. Direct computation shows that D satisfies

$$\begin{aligned} D(s_{j,n}) &= D(t_{j,n}) = D(t_i) = 0, \\ D(e_{\alpha,n}) &= \tfrac{\alpha(a)}{2\pi n} e_{\alpha,n}, \quad D(f_{\alpha,n}) = \tfrac{\alpha(a)}{2\pi n} f_{\alpha,n}, \\ D(x_\alpha) &= -\alpha(a) t y_\alpha, \quad D(y_\alpha) = \alpha(a) t x_\alpha. \end{aligned}$$

It is now clear that D is a compact operator.

The following lemma is well-known:

5.9. Lemma ([35]). *Let $\tilde{M}$ be a proper Fredholm submanifold in the Hilbert space V, and $\tilde{v} \in \nu(\tilde{M})_u$. Then $u + \tilde{v}$ is a multiplicity m focal point of $\tilde{M}$ with respect to u if and only if* 1 *is a multiplicity-m eigenvalue of the shape operator $A_{\tilde{v}}$.*

5.10. Lemma. *Let $\pi : E \to B$ be a Riemannian submersion, $v_0 \in TB_p$, $q = exp_p^B(v_0)$, and v be the normal field on $\pi^{-1}(p)$ defined by $v(x)$ being the horizontal lift of v_0 at x. Let $f : \pi^{-1}(p) \to \pi^{-1}(q)$ be the map defined by $f(x) = exp^E(v(x))$. Then f is a diffeomorphism.*

Proof. The lemma follows from the fact that for the Riemannian submersion π, the horizontal lift $\tilde{\gamma}(t)$ of a geodesic $\gamma(t)$ in B is a geodesic in E. q.e.d.

We will use the end point maps of parallel normal vector fields v of M and $\tilde{v}$ of $\tilde{M}$ in the next lemmas. Recall that these are defined to be $\eta_v : M \to G, p \to \exp(v(p))$, and $\eta_{\tilde{v}} : \tilde{M} \to V, u \to u + \tilde{v}(u)$, respectively.

5.11. Corollary. *Let M be a weakly equifocal submanifold of G, v a normal field on M, and $\tilde{v}$ the horizontal lift of v on $\tilde{M} = \phi^{-1}(M)$. Then the following hold:*

(1) $\phi \circ \eta_{\tilde{v}} = \eta_v \circ \phi$.

(2) $\phi^{-1}(M_v) = \tilde{M}_{\tilde{v}}$.

(3) Let $\tilde{y} \in \tilde{M}_{\tilde{v}}$ and $y = \phi(\tilde{y})$. Then ϕ maps $\eta_{\tilde{v}}^{-1}(\tilde{y})$ diffeomorphically to $\eta_v^{-1}(y)$

(4) If $d(\eta_{\tilde{v}})(u) = 0$ and $u \neq 0$ then $d\phi(u) \neq 0$.

Proof. By Theorem 4.5 (4), ϕ is a Riemannian submersion, (1) follows from Lemma 5.10. (2) is a consequence of (1). Let $\phi(\tilde{x}) = x$, $\tilde{x} + \tilde{v}(\tilde{x}) = \tilde{y}$. Then

$$\phi(\tilde{x} + \tilde{v}(\tilde{x})) = \phi(\eta_{\tilde{v}}(\tilde{x})) = \eta_v(\phi(\tilde{x})) = \eta_v(x) = \phi(\tilde{y}) = y,$$

which implies that ϕ maps $\eta_{\tilde{v}}^{-1}(\tilde{y})$ to $\eta_v^{-1}(y)$. Now if $\tilde{x}_1, \tilde{x}_2 \in \eta_{\tilde{v}}^{-1}(\tilde{y})$ and $\phi(\tilde{x}_1) = \phi(\tilde{x}_2) = x \in \eta_v^{-1}(y)$, then we have $\tilde{x}_1 = \tilde{x}_2$ since by Lemma 5.10 η_v maps $\phi^{-1}(x)$ diffeomorphically to $\phi^{-1}(y)$. This argument also proves that ϕ maps $\eta_{\tilde{v}}^{-1}(\tilde{y})$ onto $\eta_v^{-1}(y)$. So we have proved (3).

To see (4), suppose $d(\eta_{\tilde{v}})(u) = 0$ and $d\phi(u) = 0$ for some $u \in TM_x$. Then $u \in T(\phi^{-1}(x))_x$. But $\eta_{\tilde{v}}$ is diffeomorphic on $\phi^{-1}(x)$. So $u = 0$.

5.12. Lemma. *Let M be a submanifold in G with globally flat and abelian normal bundle, $x \in M$, $\tilde{x} \in \phi^{-1}(x)$, and $\tilde{v}$ be the horizontal lift of v by ϕ. Then v is a multiplicity-m focal normal of M with respect to x if and only if $\tilde{v}$ is a multiplicity-m focal normal of $\tilde{M}$ with respect to $\tilde{x}$. Moreover, $d\phi_{\tilde{x}}$ maps the kernel of $d\eta_{\tilde{v}}$ bijectively onto the kernel of $d\eta_v$.*

Notice that in general the eigenspace $E_\lambda(u, \tilde{v}) = ker\ d\eta_{\tilde{v}}$ is not horizontal although $d\phi$ is injective on it; see section 7 for examples.

Proof. We know from Proposition 3.7 that $q_0 = \exp(v(p_0))$ is a focal point of M with respect to p_0 if the differential of $\eta_v : M \to G$ is not injective. The multiplicity of the focal point is equal to the dimension of $\ker d\eta_v$ at p_0. By Lemma 5.9, $u_0 + \tilde{v}(u_0)$ is a focal point of $\tilde{M}$ with respect to u_0 if and only if the differential of the map $\eta_{\tilde{v}}$: $\tilde{M} \to V$ is not injective and its multiplicity is equal to the dimension of $\ker d\eta_{\tilde{v}}$ at u_0.

We therefore need to prove that the dimension of $\ker d\eta_{\tilde{v}}$ is equal to the dimension of $\ker d\eta_v$. But by Corollary 5.11, we have $\eta_v \circ \phi = \phi \circ \eta_{\tilde{v}}$. Hence we have the following commutative diagram:

$$\begin{array}{ccc} T\tilde{M}_{u_0} & \xrightarrow{d\eta_{\tilde{v}}} = & V \\ d\phi \big\downarrow & & \big\downarrow d\phi \\ TM_{p_0} & \xrightarrow{d\eta_v} & TG_{q_0} \end{array}$$

If $\tilde{X} \in \ker d\eta_{\tilde{v}}$ is nonzero, then by Corollary 5.11 (4) we have $\tilde{X} \notin \ker d\phi$. Hence $X = d\phi(\tilde{X}) \neq 0$ and $d\eta_v(X) = 0$. It follows that the dimension of $\ker d\eta_{\tilde{v}}$ is less than or equal to the dimension of $\ker d\eta_v$.

Now let $X \in \ker d\eta_v$ be a nonzero element and let $\tilde{X}$ be the horizontal lift. Then from the commutative diagram above it follows that $d\eta_v(\tilde{X}) \in \ker d\phi_{u+\tilde{v}(u)}$. By Lemma 5.10, there is an element $\tilde{Y} \in \ker d\phi_u$ such that $d\eta_{\tilde{v}}(\tilde{Y}) = d\eta_{\tilde{v}}(\tilde{X})$. Hence $d\eta_{\tilde{v}}(\tilde{X} - \tilde{Y}) = 0$. We have thus proved that the dimension of $\ker d\eta_{\tilde{v}}$ is greater or equal to the dimension of $\ker d\eta_v$. This finishes the proof.

5.13. Proof of Theorem 1.10. Lemma 5.12 proves (1), and Lemma 5.6 proves (2). It remains to prove (3). Let us first assume that M^* is weakly equifocal. Then we know from Lemma 5.8 that $\tilde{M}$ is a proper Fredholm submanifold of V. By Lemma 5.6, $\nu(\tilde{M})$ is globally flat. So to prove that $\tilde{M}$ is weakly isoparametric, it is therefore left to show that the multiplicities of the eigenvalues of $A_{\tilde{v}(u)}$ are constant and that $d\lambda(X) = 0$ for $X \in E_\lambda(\tilde{v}) = \{X \mid A_{\tilde{v}}X = \lambda X\}$. Furthermore, if M^* is equifocal, we will show that λ is constant, thereby proving that $\tilde{M}$ is isoparametric.

It follows from Lemma 5.12 that the multiplicities are constant. Let $\lambda(u)$ be a principal curvature function of $A_{\tilde{v}(u)}$. Then there is a focal point of $\tilde{M}$ in direction $\tilde{v}(u)$ at distance $\lambda(u)^{-1}$. Hence by Lemma 5.12, M has a focal point in the direction v with respect to $\phi(u) = p$, at distance $f(p) = \lambda(u)^{-1}$. Notice that $d\phi(E_\lambda(\tilde{v}))$ are the fibers of the focal distribution $\mathcal{F}_{fv}$ where $E_\lambda(\tilde{v})$ is the eigenspace of $A_{\tilde{v}(u)}$ corresponding to the eigenvalue $\lambda(u)$. Hence $df(d\phi X) = 0$ for $X \in E_\lambda(\tilde{v})$ implies $d\lambda(X) = 0$. Thus $\tilde{M}$ is weakly isoparametric. If M^* is equifocal, then f is constant. Hence λ is constant and $\tilde{M}$ therefore isoparametric.

Notice that the above arguments can also be used to prove the other direction. We therefore have that $\tilde{M}$ (weakly) isoparametric implies M^* (weakly) equifocal. This finishes the proof of the theorem.

The following theorem is proved exactly as Theorem 1.10.

5.14. Theorem. *Let $\Phi_{\underline{s}}$ be as in Corollary 4.8, and M a closed submanifold with globally flat and abelian normal bundle of $\prod_i G_{[s_{i-1},s_i]}$. Then $\Phi_{\underline{s}}^{-1}(M)$ is a (weakly) isoparametric submanifold of $H^0([0,1];\mathfrak{g})$ if and only if M is a (weakly) equifocal submanifold of $\prod_i G_{[s_{i-1},s_i]}$.*

5.15. Remark. As in the finite dimensional case, one can show that if the dimension of $E_\lambda(\tilde{v})$ is locally constant and at least two, then $d\lambda(u) = 0$ for $u \in E_\lambda(\tilde{v})$, where λ is a principal curvature function in the direction of the parallel normal vector $\tilde{v}$ of a proper Fredholm submanifold with flat normal bundle in a Hilbert space. As a consequence, it is only necessary in the definition of weakly equifocal submanifolds to assume that $df(u) = 0$ for u in the kernel of $d\eta_{fv}$ if f is a multiplicity-one focal curvature of M along a parallel normal field v. A similar remark can of course also be made on the definition of weakly isoparametric submanifolds in Hilbert spaces.

5.16. Theorem. *Let $\tilde{M}$ be a weakly isoparametric submanifold in a Hilbert space V, $\tilde{v}$ a focal normal field, and S a leaf of the focal distribution $\mathcal{F}_{\tilde{v}}$ on $\tilde{M}$. Then S is a compact Z_2-taut submanifold that is contained in a finite dimensional affine subspace of V.*

Proof. Being weakly isoparametric, $\tilde{M}$ itself is taut with respect to Z_2. It is even true that every squared distance function on $\tilde{M}$ is perfect in the sense of Bott. Set $a = \eta_{\tilde{v}}(S)$. Then S is a critical manifold of the squared distance function f_a centered at a. Since f_a satisfies condition C ([35]), S is compact.

To prove that S is taut we use the following fact proved by Ozawa in [25]: Suppose f is a perfect Morse function on $\tilde{M}$ in the sense of Bott, and S is a critical submanifold of f. If g is function on $\tilde{M}$ that restricts to a Morse function on S and has the property that $f + \delta g$ is a perfect Morse function on $\tilde{M}$ in the sense of Bott for all δ, then $g \mid S$ is a perfect Morse function on S. (Ozawa proves this in the finite dimensional case using the Morse lemma. Since the Morse Lemma is true for functions satisfying condition (C) (cf. [26]), Ozawa's result is true in infinite dimension.) Now let f_b be a squared distance function on $\tilde{M}$ that is a Morse function on S. An easy calculation shows that

$$f_a + \delta f_b = (1+\delta) f_z + c(\delta),$$

where $z = (a+\delta b)/(1+\delta)$, and $c(\delta)$ is a constant that depends only on δ. Since weakly isoparametric submanifold is taut, $f_a + \delta f_b$ is a perfect, and non-degenerate in the sense of Bott for all δ. Ozawa's result now implies that f_b is a perfect Morse function on S. This proves that S is taut.

Now exactly as when the ambient space is finite dimensional, we can show that S spans a subspace of dimension less than or equal to $n(n+3)/2$, where n is the dimension of S. To be more precise, let O_p be the osculating space of S at p, i.e., the affine space through p spanned by the first and second partial derivatives of S at p, or equivalently the affine space spanned by the the tangent space at p and the vectors $\alpha(X,Y)$ for $X, Y \in TS_p$ where α is the second fundamental form of S. It is clear that the dimension of O_p is at most $n(n+3)/2$. Now let p be the nondegenerate maximum of some squared distance function. Then the tautness of S (or even the much weaker two-piece property) implies that S is contained in O_p, cf. [8]. In particular we have proved that S spans a finite dimensional affine subspace.

5.17. Proposition. *Let M be a weakly equifocal submanifold in G, v a focal normal field, and $\mathcal{F}_v$ the focal distribution defined by v. Then the following hold:*

(1) $\mathcal{F}_v$ is integrable and its leaves are diffeomorphic to a compact, taut submanifold in a finite dimensional Euclidean space.

(2) If M is equifocal, then the leaves of $\mathcal{F}_v$ are diffeomorphic to an isoparametric submanifold in $\nu(M_v)_{\eta_v(x)}$.

Proof. The focal distribution $\mathcal{F}_v$ is integrable since it is the distribution defined as the kernel of the differential of the map η_v with constant rank. Thus the leaves of $\mathcal{F}_v$ are exactly the fiber of η_v. Let $\tilde{M}$ be the lift of M to V. Then $\tilde{M}$ is taut by Proposition 5.2. Let $\tilde{v}$ be the horizontal lift of v to a normal vector field of $\tilde{M}$. From Lemma 5.12 it follows that $\tilde{v}$ is a focal normal field since v is so. Let $x_0 \in \tilde{M}$, and $a = x_0 + \tilde{v}(x_0)$. Then we obtain Theorem 5.16 and Corollary 5.11 using

(1). If M is equifocal, then by Theorem 1.10 (3) $\tilde{M}$ is isoparametric in $V = H^0([0,1], \mathfrak{g})$. From the slice theorem for infinite dimensional isoparametric submanifolds it follows that $\eta_{\tilde{v}}^{-1}(a)$ is an isoparametric submanifold of the finite dimensional affine subspace $x_0 + \nu(\tilde{M}_{\tilde{v}})_a$ (cf. [35]). Thus Corollary 5.11 (3) yields (2).

6. Geometry of weakly equifocal submanifolds

In this section, we will give proofs for Theorem 1.6, 1.7, 1.8 and 1.9. In the following, we let (G, K) be a compact symmetric pair, $N = G/K$ the corresponding symmetric space, and $\pi : G \to N$ the Riemannian submersion associated to the right action of K on G, M a submanifold of N, and $M^* = \pi^{-1}(M)$.

We will need several lemmas for the proof of Theorem 1.9. We use the same notation as in Theorem 1.9 except that we do not assume that the normal bundle is globally flat when not explicitly stated. The first lemma is a simple consequence of the following facts:

(i) the vertical distribution $\mathcal{V}$ of π is $\mathcal{V}(g) = g\mathfrak{k}$, where $\mathfrak{k}$ is the Lie algebra of K,

(ii) $\pi(gh) = \pi(ghk)$ for $k \in K$, in particular, we have $d\pi_g(gu) = d\pi_{gk}(guk)$.

6.1. Lemma. *Let $\mathcal{H}$ denote the horizontal distribution of the Riemannian submersion $\pi : G \to N$, and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the Cartan decomposition. Then*

(1) $\mathcal{H}(g) = g\mathfrak{p}$,

(2) for $u \in \mathfrak{p}$ and $k \in K$, the horizontal lift of $d\pi_g(gu)$ at gk is guk.

Since π is a Riemannian submersion, by using Lemma 5.10 the following lemma can be proved in exactly the same way as Corollary 5.11.

6.2. Lemma. *Let v be a normal vector field on M, and v^* the horizontal lift of v to M^*. Then the following hold:*

(1) $\eta_v \circ \pi = \pi \circ \eta_{v^*}$.

(2) $\pi^{-1}(M_v) = M^*_{v^*}$.

(3) π maps $\eta_{v^*}^{-1}(y^*)$ diffeomorphically onto $\eta_v^{-1}(y)$ for any $y^* \in M^*_{v^*}$.

(4) If $d(\eta_{v^*})(u) = 0$ and $u \neq 0$, then $d\pi(u) \neq 0$.

6.3. Lemma.

(1) $\nu(M^*)$ is abelian if and only if $\nu(M)$ is abelian.

(2) *Suppose that v is a normal vector field on M, and $\nu(M)$ is abelian. Then the horizontal lift v^* of v to M^* is a parallel normal vector field on M^* if and only if v is parallel.*

(3) *$\nu(M^*)$ is globally flat and abelian if and only if $\nu(M)$ is globally flat and abelian.*

Proof. (1) Let $\mathfrak{a} = g^{-1}\nu(M)_{gK}$. Then $\exp\nu(M)_{gK} = \pi(g\exp(\mathfrak{a}))$ is contained in a flat of N if and only if $\mathfrak{a}$ is abelian. On the other hand, $\nu(M^*)_g = g\mathfrak{a}$ and hence $\exp\nu(M^*)_g = g\exp(\mathfrak{a})$ is contained in a flat of G if and only if $\mathfrak{a}$ is abelian.

(2) Let X be a vertical tangent vector. Since the covariant derivative $\nabla^*_X v^*$ only depends on v^* along a vertical curve with tangent vector X, we may assume that both v^* and X are right invariant. So

$$\nabla^*_X v^* = -\frac{1}{2}[X, v^*].$$

Now it follows from Proposition 3.1 that $\nabla^*_X v^*$ is a tangent vector of M^* since the normal bundle is abelian by (1). We have therefore shown that $\nabla^{*\perp}_X v^* = 0$ for every vertical tangent vector X.

It is only left to show that $\nabla^{*\perp}_X v^* = 0$ for every horizontal tangent vector X if and only if $\nabla^{\perp}_{d\pi X} v = 0$.

Let X be a horizontal tangent vector of M^*. We decompose $\nabla^*_X v^*$ into horizontal and vertical components:

$$\nabla^*_X v^* = (\nabla^*_X v^*)_h + (\nabla^*_X v^*)_v.$$

Since $(\nabla^*_X v^*)_h$ is the horizontal lift of $\nabla_{d\pi X} v$, $(\nabla^*_X v^*)_h$ lies in the tangent space of M^* if and only if $\nabla_{d\pi X} v$ lies in the tangent space of M. But vertical vectors are tangent to M^*. This implies that $\nabla^{*\perp}_X v^* = 0$ if and only if $\nabla^{\perp}_{d\pi X} v = 0$

(3) This is now an immediate consequence of (1) and (2).

Now the proof of Lemma 5.12 carries over to our present sitution, so we have the following lemma.

6.4. Lemma. *Let M in N be a submanifold with globally flat and abelian normal bundle, $v \in \nu(M)_x$, $x^* \in \pi^{-1}(x)$, v^* be the horizontal lift of v at x^*, and $M^* = \pi^{-1}(M)$. Then v is a multiplicity-m focal normal of M with respect to x if and only if v^* is a multiplicity-m focal normal of M^* with resptect to x^*. Morover, $d\pi_{x^*}$ maps the kernel of $d\eta_{v^*}$ bijectively onto the kernel of $d\eta_v$.*

Notice again that the kernel of $d\eta_{v^*}$ is in general not horizontal. In the next section we will discuss examples that demonstrate this.

6.5. Proof of Theorem 1.9. Lemma 6.3 proves statement (2), Lemma 6.4 proves (1), and statement (3) can be proved in a similar manner as the proof of Theorem 1.10 (3).

Applying Theorem 1.9 to the rank-one symmetric space S^{n+1}, we get

6.6. Corollary. *Let M^n be an isoparametric hypersurface of S^{n+1}. Then*

(1) M is equifocal in S^{n+1},

(2) $M^* = \pi^{-1}(M)$ is an equifocal submanifold of $SO(n+2)$, where π is the natural fibration from $SO(n+2)$ to S^{n+1}.

Similarly, if M^n is a proper Dupin hypersurface of S^{n+1}, then the lift M^* of M to $SO(n+2)$ is a weakly equifocal submanifold. If M is proper Dupin, but not isoparametric, then it follows immediately that M^* is an inhomogeneous weakly equifocal hypersurface of $SO(n+2)$. It is much more difficult to find inhomogeneous equifocal hypersurfaces in $SO(n+2)$. This will be done in the next section. Let M be an equifocal submanifold in the symmetric space N, and v a parallel normal field on M. Then the parallel set $M_v = \eta_v(M)$ is an immersed manifold since the end point map $\eta_v : M \to N$ has constant rank.

6.7. Proposition. *Let M be a compact, equifocal submanifold in a compact, symmetric space N, and v a parallel normal field. If v is not focal, then M_v is equifocal if and only if $\eta_v : M \to M_v$ is one-to-one. Moreover, if η_v is not one-to-one then*

(1) $\eta_v : M \to M_v$ is a finite cover,

(2) the normal holonomy of M_v is nontrivial, but otherwise M_v satisfies all the conditions in the definition of an equifocal submanifold.

Proof. We first prove that $\exp_x(\nu(M)_x) = \exp_{\eta_v(x)}(\nu(M_v)_{\eta_v(x)})$ if $\eta_v(x)$ is not a focal point of M with respect to x. Theorems 1.10 and 1.9 imply that the connected components of $\tilde{M} = \pi^{-1}(\phi^{-1}(M))$ are isoparametric. Therefore the corresponding statement is true for $\tilde{M}$ and $\tilde{M}_{\tilde{v}}$; see [35]. The normal spaces of $\tilde{M}$ and $\tilde{M}_{\tilde{v}}$ are the horizontal lifts of the normal spaces of M and M_v respectively which implies what we wanted to prove. It also follows that the normal bundle of M_v is abelian. By Lemma 6.3, $\nu(M_v)$ is flat if $\nu(\tilde{M}_{\tilde{v}})$ is flat. We need to show that the normal bundle of M_v has trivial holonomy if η_v is one-to-one.

Let w be a parallel normal vector field on M. Then $d\eta_v(w)$ will give rise to a globally defined parallel normal field on M_v since η_v is one-to-one. This shows that M_v is equifocal if η_v is one-to-one. If η_v is not one-to-one, let p and q be two different points in M such that $\eta_v(p) = \eta_v(q)$. Let $\alpha(t)$ be a curve connecting p and q. Then $\beta(t) = \eta_v(\alpha(t))$ is a closed curve in M_v, and $d\eta_v(v)$ will yield a parallel normal field along $\beta(t)$ that does not close up in $\beta(0) = \beta(1)$. This shows that the holonomy of the normal bundle is nontrivial. It is obvious that the focal structure of M_v is parallel.

6.8. Corollary. *Let M, N, v and η_v be as in Proposition 6.7. Then one of the following statements holds:*

(1) $\eta_v : M \to M_v$ *is a diffeomorphism,*

(2) $\eta_v : M \to M_v$ *is a finite cover,*

(3) $\eta_v : M \to M_v$ *is a fibration, and M_v is a focal submanifold of M.*

6.9. Remark. As mentioned in the introduction, if N is a real projective space and M a distance sphere in N, then M is equifocal. Its parallel manifolds are spheres, a projective hyperplane and a point. The projective hyperplane is not equifocal since it does not have trivial normal holonomy. As a parallel submanifold M doubly covers the projective hyperplane by the corresponding end point map η_v. Note that the lift of M to the Hilbert space V has two connected components.

6.10. Proposition. *Let M be an equifocal submanifold of N, and v a parallel normal field on M. Then M_v is embedded. Moreover, if v_1, v_2 are two parallel normal fields on M and $M_{v_1} \cap M_{v_2} \neq \emptyset$, then $M_{v_1} = M_{v_2}$.*

Proof. A connected component of the lift $\tilde{M}_{\tilde{v}}$ to V is either an isoparametric submanifold or a focal submanifold of the isoparametric submanifold $\tilde{M}$ and hence embedded. It follows that M_v must be embedded. If two parallel manifolds of M meet without coinciding, then the same thing is true for their lifts. But we know that parallel manifolds of an isoparametric submanifolds cannot meet without coinciding.

One consequence of Propositions 6.7 and 6.10 is that M and its parallel submanifolds give rise to an 'orbit like foliation' of M. There are three types of leaves: 'principal' when M_v is equifocal, 'exceptional' when the dimension of M_v is the same as that of M but the normal holonomy is nontrivial, and 'singular' when M_v is a focal submanifold.

We will see later that we can exclude 'exceptional' leaves when the ambient space N is simply connected.

6.11. Proposition. *If N is a simply connected, compact symmetric space, and M is a connected submanifold of N, then the lift $M^* = \pi^{-1}(M)$ is connected. If furthermore G is simply connected, then $\tilde{M} = \phi^{-1}(M^*)$ is connected.*

Proof. Since $N = G/K$ is simply connected, K is connected. So the exact sequence of homotopy of the fibration $M^* \to M$ with fiber K implies that M^* is connected. We now prove that the lift $\tilde{M} = \phi^{-1}(\pi^{-1}(M))$ of M to V is connected. We know from the proof of Proposition 5.4 that $\tilde{M}$ is diffeomorphic to the path space $P(G, p^* \times M^*)$, where $M^* = \pi^{-1}(M)$ and p^* is a lift of p. Notice that $P(G, p^* \times M^*)$ fibers over M^* with fiber $\Omega_{p^*}(G)$. Hence

$$\ldots \to \pi_0(\Omega_{p^*}(G)) \to \pi_0(P(G, p^* \times M^*)) \to \pi_0(M^*).$$

Since G is simply connected we have that $\pi_0(\Omega_{p^*}(G)) = \pi_1(G) = 0$.

6.12. Corollary. *Let N be a simply connected, compact symmetric space, M a codimension-r equifocal submanifold of N, and v a parallel normal field. If v is not focal, then $\eta_v : M \to M_v$ is a diffeomorphism, and M_v is also equifocal.*

Proof. By Proposition 6.7, it suffices to prove that η_v is one to one. Suppose $x_1, x_2 \in M$ such that $\eta_v(x_1) = \eta_v(x_2) = x$. We can assume that $N = G/K$, and G is simply connected. Let $\tilde{v}$ be the horizontal lift of v to $\tilde{M}$ via $\pi \circ \phi$. By Lemmas 5.12 and 6.4, $\tilde{v}$ is a non-focal, parallel normal field on $\tilde{M}$. Since $\tilde{M}$ is connected by Proposition 6.11 and is isoparametric in V, $\eta_{\tilde{v}} : \tilde{M} \to \tilde{M}_{\tilde{v}}$ is a diffeomorphism. But by Lemmas 5.12 and 6.2, $\eta_{\tilde{v}}$ maps the fiber Y_i over x_i of $\pi \circ \phi$ diffeomorphically onto the fibers over x for $i = 1, 2$. Because $\eta_{\tilde{v}}$ is a diffeomorphism, $Y_1 = Y_2$. In particular, this proves that $x_1 = x_2$.

6.13. Theorem. *Let M be an equifocal submanifold of the symmetric space N. Then M is totally focal in N.*

Proof. Let $c(t)$ be a geodesic that meets M orthogonally at $t = 0$ and satisfies $c(1) \in M_v$ for some focal normal field v. We have to show that $c(1)$ is a focal point of M in the direction $c'(0)$. Let $\tilde{c}(t)$ be a lift to V. Then $\tilde{c}(0)$ lies in $\tilde{M} = \phi^{-1}(\pi^{-1}(M))$ and $\tilde{c}(1) \in \tilde{M}_{\tilde{v}}$. It follows from Lemmas 5.12 and 6.4 that the components of $\tilde{M}_{\tilde{v}}$ are focal submanifolds of $\tilde{M}$, and by [35], $\tilde{c}(1)$ is a focal point of $\tilde{M}$ in direction $\tilde{c}'(0)$. Hence

$c(1)$ is a focal point of M in the direction $c'(0)$ by Lemmas 5.12 and 6.4.

It is easy to see that a k-flat in a compact, rank-k symmetric space N is a closed torus. But an r-flat with $r < k$ in N need not be a closed set. The following Theorem is proved in [17]:

6.14. Theorem. *Let N be a connected, compact, symmetric space of semi-simple type equipped with the bi-invariant metric induced from the Killing form on $\mathfrak{g}$. Suppose H acts on N isometrically, and $x \in N$ is a regular point of the H-action such that $exp(\nu(H \cdot x)_x)$ is contained in some flat. Then $exp_x(\nu(H \cdot x)_x)$ is closed in N.*

This theorem implies that if M is a homogehous, equifocal submanifold of N, then $\exp(\nu(M)_x)$ is closed. The following theorem generalizes this fact to an arbitrary equifocal submanifold:

6.15. Theorem. *Let N be as in Theorem 6.14, and M a closed equifocal submanifold of N. Then $exp_x(\nu(M)_x)$ is a closed torus for all $x \in M$.*

Proof. It is evident that if we lift M to the Lie group G and prove the theorem for M^*, then it also follows for M in N. So we may assume that $N = G$. It is proved in [17] that if $\mathfrak{t}$ is the Lie algebra of a torus in G, and $\mathfrak{a}$ is a linear subspace of $\mathfrak{t}$, then $\exp(\mathfrak{a})$ is a closed torus if and only if $\exp(\mathfrak{a}^\perp)$ is also so, where $\mathfrak{a}^\perp$ is the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{t}$.

The proof of this theorem is similar to that of Theorem 6.14 in [17] with minor changes. We repeat it for sake of completeness. Set $\mathfrak{a} = \nu(M)_x$ and $A = \exp_x(\mathfrak{a})$. Assume A is not closed, and set $B = \bar{A}$. Then B is a torus, and the Lie algebra $\mathfrak{b}$ of B is abelian. Let $\mathfrak{a}_1$ denote the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{b}$, and $A_1 = \exp(\mathfrak{a}_1)$. We first prove that B is transversal to the orbit M. To see this, notice that A is orthogonal to M whenever A meets M since the parallel manifolds of M give rise to an orbit like foliation whose leaves are met orthogonally by A. More precisely, let M_a denote the parallel submanifold of M through $a \in A$. Then $a\mathfrak{a} \perp T(M_a)_a$ and $M_a = M$ if $a \in A \cap M$. Since B is the closure of A it follows that $b\mathfrak{a} \perp TM_b$ for every $b \in B \cap M$ which implies that B is transversal to M since $TB_b = b\mathfrak{b}$ contains $b\mathfrak{a}$. By transversality, $B \cap M$ is a compact submanifold of G. Next we show that $B \cap M = A_1$. In fact, one sees easily that $T(B \cap M)_b = b\mathfrak{a}_1$ for every $b \in B \cap M$. But A_1 is the integral submanifold through e of the distribution $\triangle(g) = g\mathfrak{a}_1$ defined on G. Hence $B \cap M = A_1$. Since B

and A_1 are tori, A is a torus.

6.16. Corollary. *Let N be as in Theorem 6.14, and M a closed equifocal hypersurface of N. Then every normal geodesic to M in N is a circle.*

6.17. Proof of Theorem 1.8. Let $v \in \nu(M)_x$, $\tilde{x} \in \phi^{-1}(\pi^{-1}(x))$, and $\tilde{v}$ be the horizontal lift of v by $\pi \circ \phi$. Since $\pi \circ \phi$ is a Riemannian submersion, $\exp_x(v) = \pi(\phi(\tilde{x} + \tilde{v}))$. We prove each item of the theorem seperately below:

(a) By Corollary 5.11 (3) and Lemma 6.2 (3), $\pi \circ \phi$ maps the focal leaf of $\mathcal{F}_{\tilde{v}}$ diffeomorphically onto the focal leaf of $\mathcal{F}_v$. Then (a) follows from Proposition 5.17 (2).

Part (b) is proved in Theorem 6.15.

(c) The lift $\tilde{M} = \phi^{-1}(\pi^{-1}(M))$ is an isoparametric submanifold and has therefore an affine Coxeter group W acting on its affine normal spaces; see [35]. Since $d(\pi \circ \phi)$ maps the affine normal spaces of $\tilde{M}$ isometrically onto the normal spaces of M, W acts on $\nu(M)_x$ for all $x \in M$. Then (c) (1) follows from the standard results of isoparametric submanifolds in Hilbert space. To prove (c) (2), we note that $\tilde{M} \cap (\tilde{x} + \nu(\tilde{M})_{\tilde{x}}) = \exp^V_{\tilde{x}}(W \cdot 0)$, where $\exp^V$ is the exponential map for V. So $\exp(W \cdot 0) \subset M \cap T_x$. Conversely, if $y \in T_x \cap M$, then there exists a parallel normal field v on M such that $y = \exp(v(x))$. But $y = \pi(\phi(\tilde{x} + \tilde{v}(\tilde{x}))$, where $\tilde{v}$ is the horizontal lift of v. Then $\tilde{M}_{\tilde{v}} \cap \tilde{M} \neq \emptyset$. But $\tilde{M}$ is connected and isoparametric. So $\tilde{M} = \tilde{M}_{\tilde{v}}$. Hence $\tilde{v}(x) \in W \cdot 0$ and $y \in \exp^V(W \cdot 0)$. This proves (c) (2).

(d) Using results for the isoparametric submanifold $\tilde{M}$ in V, (2) follows from (1). So we need only prove (1). Since $\exp(\nu(M)_x)$ is contained in some flat, $\exp_x$ is a local isometry from $\nu(M)_x$ onto T_x. Thus to prove (d) (1) it suffices to prove that $\exp_x$ is one to one on $\overline{D_x}$. Suppose not. Then there exist parallel normal fields $\tilde{v}_1$ and $\tilde{v}_2$ on $\tilde{M}$ such that $\exp^V(t\tilde{v}_1(\tilde{x}))$, $\exp^V(t\tilde{v}_2(x)) \in \overline{D_x}$ for all $0 \leq t \leq 1$ and $\exp_x(v_1(x)) = \exp_x(v_2(x)) = p$, where $x = \pi(\phi(\tilde{x}))$ and $v_i = d(\pi \circ \phi)(\tilde{v}_i)$. Note that $p \in M_{v_1} \cap M_{v_2}$. By Proposition 6.10, $M_{v_1} = M_{v_2}$. But $\tilde{M}_{\tilde{v}_i} = (\pi \circ \phi)^{-1}(M_{v_i})$. Since $\tilde{M}$ is connected and isoparametric, $\tilde{v}_1 = \tilde{v}_2$. Hence $v_1 = v_2$, which proves (d) (1).

Part (e) is Theorem 6.13.

(f) Item (1) is proved in Corollary 6.12. It was proved in Corollary 6.8 that if $\exp(v(a))$ is a focal point, then η_v is a fibration. By Corollary 5.11 (3) and Lemma 6.2 (3), $\pi \circ \phi$ maps $\eta_{\tilde{v}}^{-1}(\tilde{y})$ diffeomorphically onto $\eta_v^{-1}(y)$ for $y \in M_v$. But $\eta_{\tilde{v}}^{-1}(\tilde{y})$ is the slice of the isoparametric submanifold $\tilde{M}$ in V. So $\eta_{\tilde{v}}^{-1}(\tilde{y})$ is an isoparametric submanifold of $y + \nu(\tilde{M}_{\tilde{y}})_{\tilde{y}}$. This proves (2). Item (3) can be proved exactly in the same way as (c)(2).

Part (g) follows from Proposition 6.10 and the standard results for isoparametric submanifolds in V.

(h) Note that since $\tilde{M}$ is isoparametric, the distance squared function f_a on $\tilde{M}_{\tilde{v}}$ is non-degenerate in the sense of Bott and is perfect if $a \in (\pi \circ \phi)^{-1}(p)$ (cf. [35]). By the same argument as in the proof of Proposition 5.4, $\tilde{M}_{\tilde{v}}$ is diffeomorphic to $P(G, p^* \times M^*)$ and f_a corresponds to the energy functional E, where $p^* \in \pi^{-1}(p)$. Next we prove that $P(N, p \times M_v)$ and $P(G, p^* \times M^*)$ are homotopy equivalent. For this we can assume that p lies in M so that p^* lies in M^*. We have a fibration $\pi_* : P(G, p^* \times M^*) \to P(N, p \times M)$ with fiber the space of paths in the coset p^*K starting in p^*. Since the fiber is obviously contractible, $P(N, p \times M)$ and $P(G, p^* \times M^*)$ are homotopy equivalent. Since E corresponds as in the proof of Proposition 5.4 to the distance squared function of an isoparametric submanifold in V, the indices and Morse linking cycles at critical points of E are given explicitly (cf.[35]), and (1), (2) follow.

(i) It is known that $V = \cup\{\overline{D_{\tilde{x}}} \mid \tilde{x} \in \tilde{M}\}$ and it has properties analogous to (1), (2) and (3) (cf. [35]). So (i) follows from (d).

As a consequence of Theorem 1.8, we can describe the space of parallel submanifolds of an equifocal submanifold in the simply connected case.

6.18. Corollary. *Let M be an equifocal submanifold of a simply connected semi-simple symmetric space N. Then the space of parallel submanifolds of M with the quotient topology is a simplex where the boundary points correspond to the focal manifolds.*

Note that Theorem 1.6 is a special case of Theorem 1.8 when M is an equifocal hypersurface. So the only theorem that remains to prove is 1.7.

6.19. Proof of Theorem 1.7. By Theorems 1.9 (3) and 1.10, the connected components of M^* are weakly equifocal in G, and similarly the connected components of $\tilde{M}$ are weakly isoparametric in V. The proof that an infinite dimensional isoparametric submanifold is taut (cf. [35]) shows also that an infinite dimensional weakly isoparametric submanifold in V is taut. So (b) and (c) follow. (a) follows from

Proposition 5.17 (1) and Lemma 6.2 (3).

7. Inhomogeneous examples in Hilbert spaces

The main purpose of this section is to show that there exist inhomogeneous equifocal hypersurfaces in $SO(n)$ and $H^0([0,1],\mathfrak{so}(n))$ for certain numbers n. By an inhomogeneous submanifold we mean that it is not an orbit of a subgroup of the isometry group of the ambient space.

Let $S^n = SO(n+1)/SO(n)$, $\pi : SO(n+1) \to S^n$ be the natural fibration, and $\phi : H^0([0,1],\mathfrak{so}(n+1)) \to SO(n+1)$ be the parallel transport map as before. Let $M \subset S^n$ be a submanifold, $M^* = \pi^{-1}(M)$, and $\tilde{M} = \phi^{-1}(M^*)$. First, we will derive explicit formulas relating the shape operators of M, M^* and $\tilde{M}$. In order to do this, we recall some facts about Riemannian submersions. Let $\pi : E \to B$ be a Riemannian submersion, M a submanifold of B, and $M^* = \pi^{-1}(M)$. Let X and Y be vector fields on E, and define

$$N_X Y = (\nabla_{X_h} Y_h)_v + (\nabla_{X_h} Y_v)_h,$$

where the indices h and v refer to the horizontal and vertical components respectively. Then N is tensorial in X and Y. In fact, N is one of the O'Neill tensors for the Riemannian submersion π. If both X and Y are horizontal, then

$$N_X Y = -N_Y X = (\nabla_X Y)_v.$$

7.1. Proposition. *Let $\pi : E \to B$ be a Riemannian submersion, M a submanifold of B, $v \in \nu(M)_x$ and $u \in TM_x$. Let $M^* = \pi^{-1}(M)$, and v^*, u^* be the horizontal lifts of v, u at $y \in \pi^{-1}(x)$ repectively. Then*

$$A^*_{v^*} u^* = (A_v u)^* + N_{v^*} u^*,$$

where A and A^* are the shape operators of M and M^* respectively, and $(A_v u)^*$ is the horizontal lift of $A_v u$.

Proof. Since the horizontal component of ${\nabla^*}_{u^*} v^*$ coincides with the horizontal lift of $\nabla_u v$, the horizontal component of $A^*_{v^*} u^*$ is $(A_v u)^*$. The vertical component of ${\nabla^*}_{u^*} v^*$ is tangent to M^* and hence is equal to the vertical component of $-A^*_{v^*}u^*$. Since both u^* and v^* are horizontal, the vertical component of $\nabla^*{}_{u^*}v^*$ is equal to $N_{u^*}v^* = -N_{v^*}u^*$. This proves that the vertical part of $A^*_{v^*}u^*$ is equal to $N_{v^*}u^*$.

7.2. Proposition. *Let G/K be a symmetric space, and $\pi : G \to G/K$ the corresponding Riemannian submersion. If v^* and u^* are horizontal vectors in T_eG, then*

$$N_{v^*}u^* = \frac{1}{2}[v^*, u^*].$$

Proof. Since the left invariant vector fields defined by u^* and v^* are horizontal and $N_{v^*}u^*$ is a tensor, we have

$$N_{v^*}u^* = (\nabla_{v^*}u^*)_v = \frac{1}{2}[v^*, u^*]_v.$$

But that G/K is a symmetric space implies that $[v^*, u^*]$ is vertical. So we are done.

7.3. Proposition. *Let G/K be a compact, symmetric space, $\pi : G \to G/K$ the natural fibration, M a submanifold of G/K, and $M^*\pi^{-1}(M)$. Assume that $p = \pi(e) \in M$. Let $X \in TM_p, \xi \in \nu(M)_p$, X^*_h and ξ^* be the horizontal lift of X and ξ at e respectively, and X^*_v be a vertical vector at e. Let A_ξ and $A^*_{\xi^*}$ be the shape operators of M and M^* respectively. Then*

(1) $A^*_{\xi^*}X^*_h = (A_\xi X)^* + \frac{1}{2}[\xi^*, X^*_h]$,

(2) $A^*_{\xi^*}(X^*_v) = -\frac{1}{2}[\xi^*, X^*_v]$, *which is horizontal.*

Proof. (1) follows from Lemma 4.4 and Proposition 7.2. To prove (2), we first notice that $\nabla^*_{X^*_v}\xi^*$ depends only on ξ^* along a vertical curve with tangent vector X^*_v at e. By Lemma 6.1 (2), the horizontal lift of ξ along $k \in K$ is right invariant. So we may assume that both X^*_v and ξ^* are right invariant vector fields. Hence $\nabla^*_{X^*_v}\xi^* = \frac{1}{2}[\xi^*, X^*_v]$. Since the fibers of π are totally geodesic, $\nabla^*_{X^*_v}\xi^*$ is horizontal. Now let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition, and $\mathfrak{a} = \nu(M^*)_e$. Because $\mathfrak{a}$ is abelian and the metric on G is Ad-invariant, we have $[\mathfrak{a}, \mathfrak{k}] \perp \mathfrak{a}$. This implies that $[\mathfrak{a}, \mathfrak{k}] \subset TM^*_e$ and proves (2).

7.4. Theorem. *Let $\pi : SO(n+1) \to S^n$ be the natural fibration associated to S^n, $\phi : H^0([0,1], \mathfrak{so}(n+1)) \to SO(n+1)$ the parallel transport map, M a submanifold of S^n, $M^* = \pi^{-1}(M)$, and $\tilde{M} = \phi^{-1}(M^*)$. Let $\tilde{x} \in \tilde{M}$, $x^* = \phi(\tilde{x})$, and $x = \pi(x^*)$. Let ξ be a unit normal vector of M at x, and $\xi^*, \tilde{\xi}$ the horizontal lift of ξ. Let $E_\lambda(x, \xi)$ denote* *the eigenspace of* A_ξ *for the eigenvalue* λ, *and* $E^*_\lambda(x^*,\xi^*), \tilde{E}_\lambda(\tilde{x},\tilde{\xi})$ *be defined similarly. Then we have*

(1) $d\pi_{x^*}$ *maps* $E_{(\csc\theta+\cot\theta)/2}(x^*,\xi^*)$ *isomorphically onto* $E_{\cot\theta}(x,\xi)$, *and the inverse is given by* $Y \mapsto Y^*$,

$$Y^* = x^*(X_h^* + (\csc\theta - \cot\theta)[{x^*}^{-1}\xi^*, X_h^*]) \in E^*_{(\csc\theta+\cot\theta)/2}(x^*,\xi^*),$$

where $Y \in E_{\cot\theta}(x,\xi)$ *and* $X_h^* \in \mathfrak{p}$ *such that* $x^*X_h^*$ *is the horizontal lift of* Y,

(2) $d\phi_{\tilde{x}}$ *maps* $\tilde{E}_{\frac{1}{\theta}}(\tilde{x},\tilde{\xi})$ *isomorphically to* $E^*_{(\csc\theta+\cot\theta)/2}(x^*,\xi^*)$, *and it is given explicitly as follows:*

$$\tilde{Y} = \frac{\theta}{\sin\theta} g^{-1}(\cos(\theta(t-1))X_h^* - \sin(\theta(t-1))X_v^*)g \quad \in \tilde{E}_{\frac{1}{\theta}}(\tilde{x},\tilde{\xi})$$

$$\xrightarrow{d\phi} Y^* = x^*(X_h^* + (\csc\theta - \cot\theta)X_v^*) \quad \in E^*(x^*,\xi^*),$$

where $X_v^* = [{x^*}^{-1}\xi^*, X_h^*]$ *and* $g \in H^1([0,1],G)$ *satisfies the condition that* $g(1) = e$ *and* $g * \tilde{x} = \hat{0}$.

Before proving this theorem, we need the following Lemma:

7.5. Lemma. *With the same notation as in Theorem 7.4, if* $\lambda = \frac{\cot\theta+\csc\theta}{2} = \frac{1}{2}\cot\frac{\theta}{2}$ *is an eigenvalue of* $A^*_{\xi^*}$ *at* $x^* \in M^*$ *with multiplicity* m, *then* $\exp(\theta\xi^*)$ *is a focal point of* M^* *with respect to* x^* *of multiplicity* m.

Proof. We may assume that $g = e$. Let $Y^* \in E^*_\lambda$, $y^*(t)$ be a curve in M^* such that $(y^*)'(0) = Y^*$, and $v^*(t)$ be the parallel normal field along $y^*(t)$ with $v^*(0) = \theta\xi^*$. By Lemma 3.4, we have

$$P_{y^*}(1,0)d\eta^*_{\theta\xi^*}((v^*)'(0)) = (D_1(\theta\xi^*) - D_2(\theta\xi^*)A^*_{\theta\xi^*})(Y^*),$$

where $P_{y^*}(1,0)$ is the parallel translation from 1 to 0 along y^*, and η is the end point map. Using the definition of the operators D_i and the fact that S^n has only one root, we get

$$D_1(\theta\xi^*)(Y^*) = \cos\frac{\theta}{2}Y^*, \quad D_2(\theta\xi^*)(Y^*) = \frac{2}{\theta}\sin\frac{\theta}{2}Y^*.$$

This proves that $d\eta^*_{\theta\xi^*}((v^*)'(0)) = 0$.

7.6. Proof of Theorem 7.4. By left translation, it suffices to prove (1) for $x^* = e$. Suppose $A_\xi(X) = \cot\theta X$. Set $X_v^* = [\xi^*, X_h^*]$. Then by Proposition 7.3 we have

$$A^*_{\xi^*}(X_h^*) = \cot\theta\ X_h^* + \tfrac{1}{2}[\xi^*, X_h^*] = \cot\theta\ X_h^* + \tfrac{1}{2}X_v^*,$$
$$A^*_{\xi^*}(X_v^*) = -\tfrac{1}{2}[\xi^*, X_v^*] = \tfrac{1}{2}X_h^*,$$

where the last equality follows from the fact that $\mathrm{ad}(\xi^*)^2(X_h^*) = -X_h^*$. A direct computation shows that $X_h^* + (\csc\theta - \cot\theta)X_v^*$ is in $E^*_{(\csc\theta+\cot\theta)/2}$.

Conversely, suppose $A^*(Y^*) = \frac{1}{2}(\csc\theta + \cot\theta)Y^*$. Let $Y^* = Y_h^* + Y_v^*$ be the decomposition into horizontal and vertical components, and $Y = d\pi(Y^*)$. Then Y_h^* is the horizontal lift of Y at e. Using Proposition 7.3, we get

$$\begin{aligned} A^*_{\xi^*}(Y^*) &= A^*_{\xi^*}(Y_h^*) + A^*_{\xi^*}(Y_v^*) \\ &= (A_\xi(Y))^* + \tfrac{1}{2}[\xi^*, Y_h^*] - \tfrac{1}{2}[\xi^*, Y_v^*] \\ &= \tfrac{1}{2}(\csc\theta + \cot\theta)(Y_h^* + Y_v^*). \end{aligned}$$

Comparing the vertical components of both sides of the above equation yields

$$Y_v^* = (\csc\theta - \cot\theta)[\xi^*, Y_h^*].$$

Since G/K is of rank one,

$$[\xi^*, Y_v^*] = (\csc\theta - \cot\theta)[\xi^*, [\xi^*, Y_h^*]] = -(\csc\theta - \cot\theta)Y_h^*.$$

Comparing the horizontal component then gives $(A_\xi(Y))^* = (\cot\theta Y)^*$. This proves that $d\pi(Y^*) = Y \in E_{\cot\theta}$.

To prove (2), we first assume that $\tilde{x} = \hat{0}$. Recall that the parallel transport map ϕ maps focal points of $\tilde{M}$ in $H^0([0,1], \mathfrak{g})$ to focal points of M^* in G, and maps normal geodesics of $\tilde{M}$ to normal geodesics of M^*. Note also that given $v \in \nu(\tilde{M})_u$, $u + \frac{1}{\lambda}v$ is a multiplicity m focal point of $\tilde{M}$ with respect to u if and only if λ is a principal curvature of $\tilde{M}$ with multiplicity m. So as a consequence of Lemma 7.5, we have $d\phi$ maps $\tilde{E}_{\frac{1}{\theta}}$ isomorphically onto $E^*_{(\csc\theta+\cot\theta)/2}$. In fact, the inverse of $d\phi : \tilde{E}_{\frac{1}{\theta}} \dashrightarrow E^*_{(\csc\theta+\cot\theta)/2}$ can be given explicitly as follows: By Lemma 5.7, we have

$$(*) \qquad \tilde{A}_{\tilde{\xi}}(\tilde{Z}') = [\tilde{Z}, \xi^*] + \frac{1}{2}[x^{*\prime}(0), \xi^*] - A^*_{\xi^*}(x^{*\prime}(0)),$$

where $x^{*\prime}(0) = -\tilde{Z}(1)$. Now suppose $\tilde{Z}' \in \tilde{E}_{\frac{1}{\theta}}$. Then $x^{*\prime}(0) \in E^*_{(\csc\theta+\cot\theta)/2}$. By (1), there is $X \in E_{\cot\theta}$ such that $x^{*\prime}(0) = X_h^* + s(\theta)X_v^*$, where $s(\theta) = \csc\theta = -\cot\theta$. So equation $(*)$ becomes

$$\frac{1}{\theta}\tilde{Z}' = [\tilde{Z}, \xi^*] + \frac{1}{2}[X_h^* + s(\theta)X_v^*, \xi^*] - A^*_{\xi^*}(x^{*\prime}(0)).$$

Using $[\xi^*, X_h^*] = X_v^*$, $[\xi^*, X_v^*] = -X_h^*$, and $x^{*\prime}(0) \in E^*_{(\csc\theta+\cot\theta)/2}$, we get

$$\frac{1}{\theta}\,\tilde{Z}' = -[\xi^*, \tilde{Z} + X_h^* - \cot\theta\, X_v^*].$$

Set

$$\tilde{Y} = \tilde{Z} + X_h^* - \cot\theta\, X_v^*.$$

Then

$$\tilde{Y}' = -\theta[\xi^*, \tilde{Y}], \quad \tilde{Y}(1) = -\csc\theta\, X_v^*.$$

This initial value problem can be solved explicitly, and the solution is

$$\begin{aligned}\tilde{Y}(t) &= (\cos(\theta t) - \cot\theta \sin(\theta t))\, X_h^* - (\sin(\theta t) + \cot\theta\cos(\theta t))X_v^*,\\ &= -\tfrac{1}{\sin\theta}\{\sin((t-1)\theta)X_h^* + \cos((t-1)\theta)X_v^*\}.\end{aligned}$$

So

$$\tilde{X} = -\tilde{Z}' = \frac{\theta}{\sin\theta}(\cos(\theta(t-1))X_h^* - \sin(\theta(t-1))X_v^*$$

is in $\tilde{E}_{\frac{1}{\theta}}$. To summarize, we have shown that if $\pi(e) \in M$, $x^* = e$, and $\tilde{x} = \hat{0}$, then

$$\tilde{E}_{\frac{1}{\theta}} \quad \overset{d\phi_{\hat{0}}}{\longmapsto} E^*_{(\csc\theta+\cot\theta)/2} \quad \overset{d\pi_e}{\longmapsto} E_{\cot\theta}$$

are isomorphisms and are given explicitly as

$$\begin{aligned}&\tfrac{\theta}{\sin\theta}\ (\cos(\theta(t-1))X_h^* - \sin(\theta(t-1))X_v^*)\\ &\qquad \overset{d\phi}{\longmapsto} X_h^* + (\csc\theta - \cot\theta)X_v^* \overset{d\pi}{\longmapsto} X = d\pi(X_h^*).\end{aligned}$$

Next we compute the formuals relating the curvature distributions at an arbitrary point $x \in M$. Let $x^* \in \pi^{-1}(x)$ and $\tilde{x} \in \phi^{-1}(x^*)$. Choose $g \in H^1([0,1], G)$ such that $g(1) = e$ and $g * \tilde{x} = \hat{0}$. Recall that

$$\phi(g * \tilde{x}) = g(0)\phi(\tilde{x})g(1)^{-1} = g(0) = x^* = \phi(\hat{0}) = e.$$

So $x^* = g(0)^{-1}$. Because $F_g(y) = g * y$ is an isometry of $H^0([0,1], \mathfrak{g})$, we can translate the computation at $\hat{0}$ for $F_g(\tilde{M})$ to $\tilde{x}$ by F_g to obtain the formula stated in (2).

In the remainder of this section, we will prove that there are inhomogeneous isoparametric hypersurfaces in $SO(n)$ and the Hilbert space $H^0([0,1]; \mathfrak{so}(n))$ for certain n. These examples are based on the isoparametric hypersurfaces of Clifford type in spheres that were found by Ferus, Karcher, and Münzner [13].

We first descibe the Clifford examples briefly following [30]. Let

$$\mathcal{C} = \{E_1, \dots, E_{m-1}\}$$

be a system of skew $\ell \times \ell$-matrices satisfying

$$E_i E_j + E_j E_i = -2\delta_{ij} Id.$$

We call $\mathcal{C}$ a *Clifford system*. Such Clifford systems are in one-to-one correspondence with orthogonal representations of the Clifford algebra $\mathcal{C}_{m-1}$ of R^{m-1} endowed with a negative definite metric.

We say that $u, v \in R^l$ are *Clifford orthogonal* if

$$\langle u, v\rangle = \langle E_1 u, v\rangle = \cdots = \langle E_{m-1} u, v\rangle = 0.$$

The pairs (u, v) of Clifford orthogonal vectors satisfying

$$\langle u, u\rangle = \langle v, v\rangle = \frac{1}{2}$$

form a submanifold $V_2(\mathcal{C})$ in S^{2l-1}. We call $V_2(\mathcal{C})$ the *Clifford-Stiefel manifold* of $\mathcal{C}$-orthonormal 2-frames in R^l. The tubes around $V_2(\mathcal{C})$ in S^{2l-1} turn out to be isoparametric hypersurfaces, which have four distinct principal curvatures $\lambda_1 > \cdots > \lambda_4$ with multiplicities m_1, m_2, m_1, m_2, where $m_1 = m$ and $m_2 = \ell - m - 1$. Using the classification of homogeneous isoparametric hypersurfaces, one sees that these examples are inhomogeneous if $m \neq 1$, 2, or 4 and $(m, \ell) \neq (9, 16)$. (See [13] for a detailed discussion of these examples.)

Using the same notation as in [13], we set $M^+ = V_2(\mathcal{C})$, which is one of the focal manifolds in the isoparametric family. It is proved in section 5 in [13] that the set L of points x in the focal submanifold M^+ such that there are pairwise orthogonal unit normal vectors $\xi_0, \dots, \xi_3$ in $\nu(M^+)_x$ such that

$$\dim\Big(\bigcap_{i=0}^{3} \ker A_{\xi_i}\Big) \geq 3$$

is a non-empty proper subset of M^+ if the multiplicities satisfy $9 \leq 3m_1 < m_2 + 9$ and $m_1 \neq 4$, or equivalently if $6 \leq 2m < \ell + 8$ and $m \neq 4$. This condition on m and l is always satisfied if $m > 4$ except by finitely many low dimensional examples. It now follows that M^+ and its family of isoparametric hypersurfaces are inhomogeneous without using the classification of the homogeneous ones. Applying Theorems 1.9 (3) and 1.10 (3) to these Clifford examples in $S^{2\ell-1}$, we obtain many isoparametric hypersurfaces in $H^0([0,1],\mathfrak{so}(2\ell))$ and $SO(2\ell)$. In the following, we prove that these examples are not homogenous. Because classifications of equifocal homogeneous hypersurfaces in compact Lie groups and Hilbert spaces are not known, we will use the result of Ferus-Karcher-Münzner about the set L to prove the inhomogeneity. We state this more precisely in the following theorem.

7.7. Theorem. *Let M be an isoparametric hypersurface of Clifford type in $S^{2\ell-1}$ satisfying $6 \leq 2m < \ell+8$ and $m \neq 4$. Then:*

(i) $M^* = \pi^{-1}(M)$ is an inhomogeneous equifocal hypersurface in $SO(2\ell)$, where $\pi : SO(2\ell) \to S^{2\ell-1}$ is the Riemannian submersion obtained by the identification $S^{2\ell-1} = SO(2\ell)/SO(2\ell-1)$,

(ii) $\tilde{M} = \phi^{-1}(M^*)$ is an inhomogeneous isoparametric hypersurface in V, where $V = H^0([0,1];\mathfrak{so}(2\ell))$ and $\phi : V \to SO(2\ell)$ is the parallel transport map.

We will need the following simple Lemma.

7.8. Lemma. *Let $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition corresponding to $S^{n-1} = SO(n)/SO(n-1)$. Suppose $a \in \mathfrak{p}$ such that $ad(a)^2 \,|\, \mathfrak{p}\cap\mathfrak{a}^\perp = -\,id$, and $x, y, z \in \mathfrak{p}$ are orthogonal to a. If $r = [a,x] = [y,z]$ then $r = 0$ and $x = 0$.*

Proof. Let $\mathfrak{g}_C(\lambda)$ denote the eigenspace of $\mathrm{ad}(a)$ on $\mathfrak{g}_C = \mathfrak{g}\otimes C$ corresponding to the eigenvalue λ. Then $a^\perp \cap \mathfrak{p} \subset \mathfrak{g}_C(i)+\mathfrak{g}_C(-i)$. From

$$[\mathfrak{g}_C(\lambda_1), \mathfrak{g}_C(\lambda_2)] \subset \mathfrak{g}_C(\lambda_1+\lambda_2)$$

it follows that

$$[a,x] \in \mathfrak{g}_C(i) + \mathfrak{g}_C(-i)$$

and

$$[y,z] \in \mathfrak{g}_C(2i) + \mathfrak{g}_C(-2i) + \mathfrak{g}_C(0).$$

Hence $r = [a,x] = [y,z] = 0$. Since $\mathrm{ad}(a)^2 = -\,\mathrm{id}$ on $\mathfrak{a}^\perp \cap \mathfrak{p}$, $x = 0$.

7.9. Proof of Theorem 7.7. Let L denote the set of points x in the focal submanifold M^+ such that there are orthonormal vectors $\xi_1, \ldots, \xi_4$ in $\nu(M^+)_x$ satisfying

$$\dim\Big(\bigcap_{i=1}^{4} E_0(x,\xi_i)\Big) \geq 3.$$

It is proved in [13] that L is a proper subset of M^+. Let $\mathfrak{D}$ denote the set of points $\tilde{x}$ in $\tilde{M}^+ = (\pi\circ\phi)^{-1}(M^+)$ such that there are orthonormal vectors $\tilde{\xi}_1,\dots,\tilde{\xi}_4$ in $\nu(\tilde{M}^+)_{\tilde{x}}$ such that

$$\dim\Big(\bigcap_1^4(\tilde{E}_{\frac{2}{\pi}}(\tilde{x},\tilde{\xi}_i)+\tilde{E}_{-\frac{2}{\pi}}(\tilde{x},\tilde{\xi}_i)\Big)\geq 3.$$

Let $x^* = \phi(\tilde{x})$, $x = \pi(x^*)$, $\xi_i^* = d\phi(\tilde{\xi}_i)$ and $\xi_i = d\pi(\xi_i^*)$. Since both π and ϕ are Riemannian submersion, $\{\xi_1^*,\dots,\xi_4^*\}\subset\nu(M^{+^*})_{x^*}$ and $\{\xi_1,\dots,\xi_4\}\subset\nu(M^+)_x$ are orthonormal.

Choose $g\in H^1([0,1],G)$ such that $g(1)=e$ and $g*\tilde{x}=\hat{0}$. We claim that

$$\begin{aligned}&\cap_1^4\Big(\tilde{E}_{\frac{2}{\pi}}(\tilde{x},\tilde{\xi}_i)+\tilde{E}_{-\frac{2}{\pi}}(\tilde{x},\tilde{\xi}_i)\Big)\\ &=\Big\{\sin(\tfrac{\pi t}{2})g^{-1}X_h^*g\,|\,x^*X_h^* \text{ is the horizontal lift of } X\in\cap_1^4E_0(x,\xi_i)\Big\}.\end{aligned}$$

In particular, this shows that $\mathfrak{D}=\phi^{-1}(\pi^{-1}(L))$. By definition of $\mathfrak{D}$, $\tilde{M}^+$ being homogeneous would imply $\mathfrak{D}$ is either equal to $\tilde{M}^+$ or an empty set. Since L is a proper non-empty subset of M^+, $\mathfrak{D}$ is a proper subset of $\tilde{M}^+$. Hence $\tilde{M}^+$ is inhomogeneous. To prove our claim, we first note that by Theorem 7.4 any vector in $\tilde{E}_{\frac{2}{\pi}}(\tilde{x},\tilde{\xi})$ is of the form

$$g\Big(\sin\frac{\pi t}{2}X_h^*+\cos\frac{\pi t}{2}[\xi^*,X_h^*]\Big)g^{-1}$$

for some $X\in E_0(x,\xi)$, and any vector in $\tilde{E}_{\frac{-2}{\pi}}(\tilde{x},\tilde{\xi})$ is of the form

$$g\Big(\sin\frac{\pi t}{2}Y_h^*-\cos\frac{\pi t}{2}[\xi^*,Y_h^*]\Big)g^{-1}$$

for some $Y\in E_0(x,\xi)$. Therefore a typical element in $\tilde{E}_{\frac{2}{\pi}}(\tilde{x},\tilde{\xi})+\tilde{E}_{-\frac{2}{\pi}}(\tilde{x},\tilde{\xi})$ is of the form

$$g\Big(\sin\frac{\pi t}{2}(X+Y)_h^*+\cos\frac{\pi t}{2}[\xi^*,(X-Y)_h^*]\Big)g^{-1}.$$

Now suppose $Z\in\cap(\tilde{E}_{\frac{2}{\pi}}(\tilde{x},\tilde{\xi}_i)+\tilde{E}_{-\frac{2}{\pi}}(\tilde{x},\tilde{\xi}_i))$. Then there exist $X(i),Y(i)\in E_0(x,\xi_i)$ for $i=1,\dots,4$ such that

$$Z=g\Big(\sin(\frac{\pi t}{2})X(i)_h^*+\cos(\frac{\pi t}{2})[x^{*-1}\xi_i^*,Y(i)_h^*]\Big)g^{-1},$$

where $x^*X(i)^*_h$ and $x^*Y(i)^*_h$ are the horizontal lifts of $X(i)$ and $Y(i)$ at x^* respectively. Since Z is continuous and $x^* = g(0)$, $Z(0) = x^*[x^{*-1}\xi_i^{*-1}, Y(i)^*_h]x^{*-1}$. This implies that

$$[x^{*-1}\xi_i^*, Y(i)^*_h] = [x^{*-1}\xi_j^*, Y(j)^*_h]$$

for any $1 \leq= i, j \leq 4$. Next we claim that $Z(0) = 0$. To see this, we may assume $x^* = e$. $\xi_1^*, \ldots, \xi_4^*$ are perpendicular to $E^*_{\pm\frac{1}{2}}(x^*, \xi_i^*)$ since they are in $\nu(M^*)_e$. Now let $a = \xi_1^*$, $x = Y(1)^*_h$, $y = \xi_2^*$ and $z = Y(2)^*_h$ in Lemma 7.8. It then follows that $r = Z(0) = [\xi_1^*, Y(1)^*_h] = 0$ and $Y(1)^*_h = 0$. So

$$\begin{aligned} Z &= g\Big(\sin(\tfrac{\pi t}{2})X(i)^*_h + \cos(\tfrac{\pi t}{2})[x^{*-1}\xi_i^*, Y(i)^*_h]\Big)g^{-1} \\ &= g\Big(\sin(\tfrac{\pi t}{2})X(i)^*_h\Big)g^{-1}. \end{aligned}$$

Moreover,

$$Z(1) = X(1)^*_h = \ldots = X(4)^*_h.$$

Hence $x^*Z(1) = X^*_h$, the horizontal lift of some $X \in \bigcap_1^4 E_0(x, \xi_i)$, and

$$Z = \sin\frac{\pi t}{2} g X^*_h g^{-1}.$$

This proves our claim and completes the proof of (2).

Now suppose $M^{+*} = \pi^{-1}(M)$ is the orbit of some subgroup H of $G \times G$ through $x^* = e^a$, where $G = SO(2\ell)$. It is proved in [37] that $\tilde{M}^+$ is then the orbit of the isometric action of

$$P(G, H) = \{g \in H^1([0,1], G) \mid (g(0), g(1)) \in H\}$$

through the constant path $\hat{a}$, a contradiction. This proves (1).

8. Open problems

1. Suppose M is an equifocal submanifold of a simply connected, compact symmetric space N of codimension $r \geq 2$ such that the action of the associated affine Weyl group on $\nu(M)_p$ is irreducible. Is M homogeneous, i.e., is M an orbit of some hyperpolar action on N? This is true for irreducible equifocal (i.e., isoparametric) hypersurfaces in Euclidean spaces if the codimension is at least three; see [39].

2. By Theorem 1.6, the normal geodesic γ_x of an equifocal hypersurface in a compact, semi-simple symmetric space N is closed and there are $2g$ focal points in γ_x. Is there a generalization of Münzner's theorem for isoparametric hypersurfaces in spheres saying that $g = 1$, 2,= 3, 4 or 6? Wu proved in [42] that if N is a complex or quaternionic projective space, then $g = 1, 2$ or 3.

3. Can $H_*(M, Z_2)$ be computed explicitly in terms of the associated affine Weyl group and multiplicities?

4. If M is an irreducible, codimension $r \geq 2$ isoparametric submanifold of an infinite dimensional Hilbert space, is M homogeneous?

5. Let $\triangle_y$ be defined as in Theorem1.8. Is there a finite group acting on the normal torus T_x of an equifocal submanifold in a simply connected symmetric space that is simply transitive on the set of chambers $\{\triangle_y \mid y \in T_x \cap M\}$?

6. Is there a similar theory for equifocal submanifolds in simply connected, non-compact, symmetric spaces?

7. Lie sphere geometry of S^n (see [29], [7]) can be naturally extended to compact symmetric spaces. To be more precise, let N be a simply connected, compact symmetric space, and the unit tangent bundle T^1N be equipped with the natural contact structure. Given an immersed Legendre submanifold $f : X \to T^1N$ and $t \in R$, let $f_t : X \to T^1N$ denote the map $f_t(u) = d\exp_{tu}(tu)$. We call λ a *multiplicity-m focal radius* of X along u_0 if f_λ is singular at u_0 and the dimension of the kernel of $d(f_t)_{u_0}$ is m. A connected k-dimensional submanifold S of X is called *a focal leaf* of X if there exists a smooth function $\lambda : S \to R$ such that $\lambda(u)$ is a multiplicity k focal radius of X along u and $\ker(d(f_{\lambda(u)})_u) = TS_u$ for all $u \in S$. A Legendre submanifold X of T^1N is called *Dupin* if every focal leaf projects down to an intersection of geodesic spheres in N. A Dupin Legendre submanifold is called *proper* if the focal radii have constant multiplicities. Note that if M is an immersed submanifold of N, then the unit normal bundle $\nu^1(M)$ is an immersed Legendre submanifold of T^1N. Moreover, a hypersurface M is weakly equifocal if and only if $\nu^1(M)$ is a proper, Dupin Legendre submanifold of T^1N. It follows from the results of this paper that Dupin Legendre submanifolds of T^1N share many of the same properties as Dupin Legendre submanifolds (these are called Dupin Lie geometric hypersurfaces in [29]) of T^1S^n. So the following questions arise naturally:

(a) What is the group of diffeomorphisms $g : N \to N$ such that the induced map $dg : T^1N \to T^1N$ maps Dupin Legendre submanifolds to Dupin Legendre submanifolds? When $N = S^n$, this group is the group of conformal diffeomorphisms (Möbius transforamtions).

(b) What is the group of contact transformations of T^1N that map Dupin Legendre submanifolds to Dupin Legendre submanifolds? Or equivalently, what is the group of contact transformations of T^1N that maps Legendre spheres to Legendre spheres? Here a Legendre sphere in T^1N is defined to be either the unit normal bundle of a geodesic hypersphere or a fiber of the projection $\pi : T^1N \to N$. When $N = S^n$, this group is the group of Lie sphere transformations, which is isomorphic to $O(n+1,2)/Z_2$ and is generated by conformal transformations and the parallel translations f_t (cf. [7], [29]).

(c) A compact submanifold M in a compact symmetric space N is called *taut* if for generic $p \in N$ the energy functional $E : P(N, p \times M) \to R$ is a perfect Morse function. Is a taut submanifold Dupin? Is tautness invariant under the transformations in questions (a) and (b)?

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